$$b_r = \sum_{t \geq 0} \begin{bmatrix} r+2t \\ t \end{bmatrix} a_{r+2t}$$

$$= \sum_{t \geq 0} \begin{bmatrix} r+2t \\ t \end{bmatrix} \sum_{h \geq 0} T(r+2t, h)\, b_{r+2t+h}$$

$$= \sum_{t, h \geq 0} \begin{bmatrix} r+2t \\ t \end{bmatrix} T(r+2t, h)\, b_{r+2t+h}$$

We require

$$T(r, 0) = 1.$$

Let $N = 2t+h > 0$ so that $h = N - 2t$.

We require

(**)' $$\sum_{0 \leq 2t \leq N} \begin{bmatrix} r+2t \\ t \end{bmatrix} T(r+2t, N-2t) = 0$$

for $N > 0$. We want

(***) $$\begin{cases} T(r, 0) = 1, \\ T(r, 1) = 0, \quad \text{(by letting } N=1) \\ \text{and} \\ T(r, N) = - \sum_{1 \leq 2j \leq N} \begin{bmatrix} r+2j \\ j \end{bmatrix} T(r+2j, N-2j). \end{cases}$$

(***) uniquely define $T(r, n)$ for all $r, n \geq 0$.

It is clear that

$$T(r, N) = 0 \qquad \text{if } N \text{ is odd}.$$