It seems that

Theorem: If $T(r,n)$ satisfies $(**)'$ Then we have:

(1) $T(r, 2n+1) = 0$

(2) $T(r, 2n) = (-1)^n q^{n(n-1)/2} \dfrac{(q^{r+1})_{n-1}(1-q^{r+2n})}{(q)_n}$

Proof:

From p.11 of Notes Ch. 4,

$$\sum_{j=0}^{\infty} \frac{(a)_j (b)_j t^j}{(c)_j (q)_j} = \frac{(c/b)_\infty (bt)_\infty}{(c)_\infty (t)_\infty} \sum_{m=0}^{\infty} \frac{\left(\frac{atb}{c}\right)_m (b)_m (c/b)^m}{(bt)_m (q)_m}$$

(by Heine's transf).

Let $b = q^{-n}$.

$$\frac{(c/b)_\infty}{(c)_\infty} = \frac{(q^n c)_\infty}{(c)_\infty} = \frac{1}{(c)_n}$$

$$\frac{(bt)_\infty}{(t)_\infty} = \frac{(q^{-n}t)_\infty}{(t)_\infty} = (1-q^{-n}t)(1-q^{-n+1}t)\cdots(1-q^{-1}t)$$

$$= (tq^{-n})_n.$$

Hence

$$\sum_{j=0}^{\infty} \frac{(a)_j (q^{-n})_j t^j}{(c)_j (q)_j} = \frac{(tq^{-n})_n}{(c)_n} \sum_{j\geq 0} \frac{(q^{-n})_j (atq^{-n}/c)_j (cq^n)^j}{(q^{-n}t)_j (q)_j}$$

$$\sum_{j=0}^{n} \frac{(a)_j (q^{-n})_j t^j}{(c)_j (q)_j} = \frac{1}{(C)_n} \sum_{j=0}^{n} \frac{(atq^{-n}/c)_j}{(q)_j} (tq^{-n+j})_{n-j} (q^{-n})_j \cdot (cq^n)^j$$

(since $(q^{-n})_j = 0$ if $j > n$)