$$\sum_{0\le 2j\le 2n} \begin{bmatrix} r+2j \\ j \end{bmatrix} \tau(r+2j,\, 2n-2j)$$

$$= \sum_{j=0}^{n} \begin{bmatrix} r+2j \\ j \end{bmatrix} \frac{(-1)^{n-j} q^{(n-j)(n-j-1)/2} (q^{r+2j+1})_{n-j-1} (1-q^{r+2n})}{(q)_{n-j}}$$

$$= \sum_{j=0}^{n} \frac{(-1)^{n-j} q^{(n-j)(n-j-1)/2} (q)_{r+2j} (q^{r+2j+1})_{n-j-1} (1-q^{r+2n})}{(q)_j (q)_{r+j} (q)_{n-j}}$$

$$= \sum_{j=0}^{n} \frac{(-1)^{n-j} q^{(n-j)(n-j-1)/2} (q)_{r+j+n-1} (1-q^{r+2n})}{(q)_j (q)_{r+j} (q)_{n-j}}$$

[Recall $(q^{m+1})_j (q)_m = (q)_{m+j}$

and

$$(q^{-n})_j = \frac{(q)_n}{(q)_{n-j}} (-1)^j q^{j(j-1)/2 - nj}$$

$$\& \quad \frac{1}{(q)_{n-j}} = \frac{(q^{-n})_j}{(q)_n \, (-1)^j q^{j(j-1)/2 - nj}} \Bigg]$$

$$= \sum_{j=0}^{n} \frac{(-1)^n (1-q^{r+2n}) q^{n(n-1)/2+j} (q^{r+n})_j (q^{-n})_j (q)_{r+n-1}}{(q)_j (q^{r+1})_j (q)_r (q)_n}$$

$$= \frac{(-1)^n (1-q^{r+2n}) q^{n(n-1)/2} (q)_{r+n-1}}{(q)_r (q)_n} \sum_{j=0}^{n} \frac{(q^{-n})_j (q^{r+n})_j q^j}{(q^{r+1})_j (q)_j}$$