Hence,

$$a_r = \frac{q^{r^2/2}}{(q)_r} \cdot \sum_{j=0}^{\infty} \frac{q^{j^2+(r+1)j-j}}{(q)_j}$$

and

$$b_0 = \frac{1}{(q)_\infty}$$

$$b_r = \frac{q^{r^2/2}}{(q)_\infty} \quad \text{for } r \geq 1.$$

So $\quad b_r = \dfrac{q^{r^2/2}}{(q)_\infty} \quad$ for $r \geq 0$.

Now $\quad a_0 = \displaystyle\sum_{j=0}^{\infty} \frac{q^{j^2}}{(q)_j}.$

$$a_0 = \sum_{h \geq 0} T(0,h)\, b_h = \sum_{h \geq 0} T(0,2h)\, b_{2h}$$

Hence,

$$\sum_{j=0}^{\infty} \frac{q^{j^2}}{(q)_j} = \frac{1}{(q)_\infty} + \sum_{h \geq 1} (-1)^h q^{h(h-1)/2} \frac{(q)_h}{(q)_h} \frac{(1-q^{2h})}{(1-q^h)} \cdot \frac{q^{2h^2}}{(q)_\infty}$$

$$= \frac{1}{(q)_\infty}\left(1 + \sum_{h \geq 1} (-1)^h q^{h(5h-1)/2} (1+q^h)\right)$$

$$= \frac{1}{(q)_\infty}\left(1 + \sum_{h \geq 1} (-1)^h q^{h(5h-1)/2} + \sum_{h \geq 1} (-1)^h q^{h(5h+1)/2}\right)$$