Theorem

$$\alpha_n = \begin{cases} 1 & \text{if } n=0 \\ (-1)^n q^{n(n-1)/2}(1+q^n) & \text{if } n\geq 1 \end{cases}$$

$$\beta_n = \begin{cases} 1 & \text{if } n=0 \\ 0 & \text{if } n\geq 1 \end{cases}$$

is a Bailey pair with $a=1$.

Proof: We need

$$(z)_N = \sum_{j=0}^{N} (-1)^j z^j q^{j(j-1)/2} \begin{bmatrix} N \\ j \end{bmatrix}$$

(See p. 36 of TEXT or p. 6 of Notes for Ch. 4).
We have to show that

$$\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(q)_{n-r}(q)_{n+r}}.$$

Clearly true when $n=0$ since $\beta_0=\alpha_0=1$.
Let $n\geq 1$. Then

$$\sum_{j=0}^{n} \frac{\alpha_j}{(q)_{n-j}(q)_{n+j}} = \frac{1}{(q)_n^2} + \sum_{j=1}^{n} \frac{(-1)^j q^{j(j-1)/2}(1+q^j)}{(q)_{n-j}\,(q)_{n+j}}$$

$$= \sum_{j=-n}^{n} \frac{(-1)^j q^{j(j-1)/2}}{(q)_{n-j}\,(q)_{n+j}}$$