Therefore, $$\sum_{j=0}^{\infty} q^{j^2}\beta_j' = \frac{1}{(q)_\infty}\sum_{n=0}^{\infty} q^{n^2}\alpha_n'. \tag{40}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n} = \frac{1}{(q)_\infty}\left(1+\sum_{n\geq 1}(-1)^n q^{n(5n-1)/2}(1+q^n)\right)$$

(as before).

$$= \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})}$$

Iterating Further

$$\alpha_n'' = q^{n^2}\alpha_n' = \begin{cases} 1 & \text{if } n=0 \\ (-1)^n q^{n(5n-1)/2}(1+q^n) & \text{if } n\geq 1 \end{cases}$$

$$\beta_n'' = \sum_{j=0}^{n} \frac{q^{j^2}}{(q)_{n-j}}\beta_j = \sum_{j=0}^{n} \frac{q^{j^2}}{(q)_{n-j}(q)_j}$$

forms a Bailey pair.

Hence $$\sum_{n=0}^{\infty} q^{n^2}\beta_n'' = \frac{1}{(q)_\infty}\sum_{n=0}^{\infty} q^{n^2}\alpha_n''$$

$$\sum_{n=0}^{\infty}\sum_{j=0}^{n} \frac{q^{n^2+j^2}}{(q)_{n-j}(q)_j} = \frac{1}{(q)_\infty}\left(1+\sum_{n\geq 1}(-1)^n q^{n(7n-1)/2}(1+q^n)\right).$$

$$= \prod_{\substack{n\geq 1 \\ n\not\equiv 0,\pm 3 \pmod 7}} \frac{1}{(1-q^n)}$$