Let $C_m(n)$ = # of ptns of n enumerated by $C(n)$

(i.e. ptns $\pi = (\pi_1, \pi_2, \ldots, \pi_k)$

where $\pi_i - \pi_{i+1} \geq 3$ &

$\pi_i - \pi_{i+1} > 3$ if $3 | \pi_i$)

and the largest part $\pi_1 \leq m$.

$C_m(n) - C_{m-1}(n)$ = # of ptns of n enumerated by $C(n)$ with largest part $\pi_1 = m$

Case 1. $m \equiv 1$ or $2 \pmod 3$

We remove the largest part and obtain a partition of $n-m$ whose largest part $\leq m-3$ and satisfies the difference conditions.

Case 2. $m \equiv 0 \pmod 3$

We remove the largest part and obtain a partition of $n-m$ whose largest part $\leq m-4$ and satisfies the difference condition.

Hence

$$C_m(n) - C_{m-1}(n) = \begin{cases} C_{m-3}(n-m) & \text{if } m \equiv 1, 2 \pmod 3 \\ C_{m-4}(n-m) & \text{if } m \equiv 0 \pmod 3 \end{cases}$$

So we have

$$C_{3m+1}(n) = C_{3m}(n) + C_{3m-2}(n-3m-1)$$
$$C_{3m+2}(n) = C_{3m+1}(n) + C_{3m-1}(n-3m-2)$$
$$C_{3m+3}(n) = C_{3m+2}(n) + C_{3m-1}(n-3m-3)$$

We let

$$f(q) = \sum_{n=0}^{\infty} C(n) q^n \quad \& \quad f_m(q) = \sum_{n \geq 0} C_m(n) q^n.$$