$$c_{3m+1}(n)q^n = c_{3m}(n)q^n + q^{3m+1}c_{3m-2}(n-3m-1)q^{n-3m-1}$$

$$\sum_{n=0}^{\infty} c_{3m+1}(n)q^n = \sum_{n=0}^{\infty} c_{3m}(n)q^n + q^{3m+1}\sum_{n\geq 3m+1} c_{3m-2}(n-3m-1)q^{n-3m-1}$$

$$= \sum_{n=0}^{\infty} c_{3m}(n)q^n + q^{3m+1}\sum_{n=0}^{\infty} c_{3m-2}(n)q^n$$

Hence

$$f_{3m+1}(q) = f_{3m}(q) + q^{3m+1}f_{3m-2}(q).$$

Similarly,

$$f_{3m+2}(q) = f_{3m+1}(q) + q^{3m+2}f_{3m-1}(q)$$

$$f_{3m+3}(q) = f_{3m+2}(q) + q^{3m+3}f_{3m-1}(q).$$

Let $\alpha_m(q) := f_{3m+2}(q)$.

So

$$f_{3m+3}(q) = \alpha_m(q) + q^{3m+3}\alpha_{m-1}(q),$$

$$f_{3m+1}(q) = \alpha_m(q) - q^{3m+2}\alpha_{m-1}(q),$$

and

$$\alpha_m - q^{3m+2}\alpha_{m-1} = \alpha_{m-1} + q^{3m}\alpha_{m-2} + q^{3m+1}(\alpha_{m-1} - q^{3m-1}\alpha_{m-2})$$

Hence,

(*) $$\alpha_m = (1+q^{3m+1}+q^{3m+2})\alpha_{m-1} + q^{3m}(1-q^{3m})\alpha_{m-2}$$

for $m \geq 2$. This recurrence together with the initial conditions

$$\cancel{\alpha_0(q) = f_2(q)}$$

$$\alpha_0 = f_2(q) = 1+q+q^2$$