$$\sum_{n=0}^{\infty} s_n z^n - \sum_{n=0}^{\infty} s_n z^{n+1} = \sum_{n=0}^{\infty} s_n q^{3n} z^n$$
$$+ \sum_{n=0}^{\infty} z^{n+1}(q^{3n+1}+q^{3n+2}) s_n$$
$$+ \sum_{n=0}^{\infty} z^{n+2} q^{3n+3} s_n$$

$$\sum_{n=0}^{\infty} s_n z^n - \sum_{n=0}^{\infty} s_{n-1} z^n$$
$$= \sum_{n=0}^{\infty} z^n \left( q^{3n} s_n + (q^{3n-1}+q^{3n-2}) s_{n-1} + q^{3n-3} s_{n-2} \right)$$

(assuming $s_{-1}=s_{-2}=0$).

Hence

$$(**) \quad (1-q^{3n}) s_n = (1+q^{3n-1}+q^{3n-2}) s_{n-1} + q^{3n-3} s_{n-2}$$

for $n \geq 2$. (also true for $n \geq 0$ assuming $s_{-1}=s_{-2}=0$).

~~Let~~ $\cancel{S_n = (q^3;q^3)_n s_n}$

$$\cancel{S_{n-1} = (q^3;q^3)_{n-1} s_{n-1} = \frac{(q^3;q^3)_n s_{n-1}}{1-q^{3n}}}$$

$$\cancel{S_{n-2} = (q^3;q^3)_{n-2} s_{n-2} = \frac{(q^3;q^3)_{n-1} s_{n-2}}{1-q^{3n-2}}}$$

We multiply both sides of (**) by $(q^3;q^3)_{n-1}$

$$(q^3;q^3)_n s_n = (1+q^{3n-1}+q^{3n-2})(q^3;q^3)_{n-1} + q^{3n-3}(1-q^{3n-3}) \cdot (q^3;q^3)_{n-2} s_{n-2}$$