Thus
$$S_n = (1+q^{3n-1}+q^{3n-2})\,S_{n-1} + q^{3n-3}(1-q^{3n-3})\,S_{n-2}$$

where $S_n = (q^3;q^3)_n\, s_n$,

and

S_{m+1} satisfies the same recurrence (*) as α_m.

But $S_1 = (1-q^3)\,s_1 = (1+q+q^2)\,s_0 = 1+q+q^2$ (by **)

$S_1 = \alpha_0$

and $S_0 = 1 = \alpha_{-1}$.

It follows that $S_{m+1} = \alpha_m$ for $m \geq -1$.

Hence,
$$s_n = \frac{S_n}{(q^3;q^3)_n} = \frac{\alpha_{n-1}}{(q^3;q^3)_n} \quad \text{for } n \geq 0,$$

and
$$\frac{(-zq;q^3)_\infty\,(-zq^2;q^3)_\infty}{(z;q^3)_\infty} = \sum_{n=0}^{\infty} \frac{\alpha_{n-1}(q)\, z^n}{(q^3;q^3)_n}.$$

We need the following

Lemma: Suppose $\sum_{n=0}^{\infty} a_n z^n$ converges for $|z|<1$. Suppose $a_n \geq 0$ for $n \geq 0$ & $\lim_{n\to\infty} a_n = a$. Then
$$\lim_{z\to 1^-} (1-z) \sum_{n=0}^{\infty} a_n z^n = a.$$