(11)

$$(1-z)^{-1} = \frac{1}{1-z} = \sum_{m=0}^{\infty} z^m \quad (\text{for } |z|<1)$$

$$\frac{z}{1-z} = \sum_{m=1}^{\infty} m z^m$$

$$\frac{z}{1+z} = \sum_{m=1}^{\infty} m(-1)^{m+1} z^m$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{(-1)^n q^n}{(1+q^n)^2} = \sum_{n=1}^{\infty} (-1)^n \sum_{m=1}^{\infty} m(-1)^{m+1} q^{nm}$$

$$= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} m(-1)^{n+m+1} q^{nm}$$

It follows that

$$r_4(N) = 8 \sum_{\substack{n,m\geq 1 \\ nm=N}} m(-1)^{n+m+1+nm}$$

$$= 8 \sum_{d|N} d\,(-1)^{1+N+d+\frac{N}{d}}$$