Case 1 N is odd. Then if $d|N$, d is odd

Then $(-1)^{1+N+d+N/d} = 1$

and

$$r_4(N) = 8 \sum_{d|N} d$$

Case 2 N is even, $N = m2^{\alpha}$ (m odd, $\alpha \geq 1$).

Any divisor (+ve) d' of N can be written uniquely as $d' = d2^j$ where $d|m$ & $0 \leq j \leq \alpha$.

Hence

$$r_4(N) = 8 \sum_{d'|N} d' (-1)^{1+N+d'+N/d'}$$

$$= 8 \sum_{d|m} \sum_{j=0}^{\alpha} d2^j (-1)^{1+N+d2^j+\cancel{N/d}\left(\frac{m}{d}\right)2^{\alpha-j}}$$

$$= 8 \sum_{d|m} d \left(1 + \sum_{j=1}^{\alpha-1} 2^j(-1) + 2^{\alpha}\right)$$

$$= 8 \sum_{d|m} d(1+2) \qquad = 8 \sum_{\substack{d|N \\ 4\nmid d}} d$$

Since $1+2+\cdots+2^{\alpha-1} = 2^{\alpha}-1$

& $2+\cdots+2^{\alpha-1} = (2^{\alpha}-2)$.

Hence we have

Theorem (Jacobi) For $n \geq 1$,

$$r_4(n) = 8 \sum_{\substack{d|n \\ 4\nmid d}} d$$