Hence we have

$$\sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\frac{q^n}{(1-q^n)^2} = \sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\sum_{m=1}^{\infty} m q^{mn}$$

$$= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{n}{5}\right) m q^{mn}$$

$$= \sum_{N=1}^{\infty}\left(\sum_{\substack{nm=N \\ n,m\geq 1}}\left(\frac{n}{5}\right) m\right) q^N$$

$$= \sum_{N=1}^{\infty}\left(\sum_{d\mid N}\left(\frac{d}{5}\right)\frac{N}{d}\right) q^N$$

Here, $\left(\frac{n}{5}\right) = \begin{cases} 0 & \text{if } n\equiv 0 \pmod 5 \\ 1 & \text{if } n\equiv 1,4 \pmod 5 \\ -1 & \text{if } n\equiv 2,3 \pmod 5 \end{cases}$

is the Legendre symbol mod 5.

Let $a_t(n)$ = # of partitions of n that are t-cores (i.e. have no hooknumbers that are multiples of t).

Then recall that

$$\sum_{n=0}^{\infty} a_t(n) q^n = \frac{(q^t;q^t)_\infty^t}{(q)_\infty} \quad \text{for } |q|<1.$$

Hence,

$$\sum_{n=0}^{\infty} a_5(n) q^{n+1} = \frac{q\,(q^5;q^5)_\infty^5}{(q)_\infty}$$