$$(azq^{1-N})_\infty = (1-azq^{1-N})\cdots(1-azq^{-1})(az)_\infty$$

$$= (-azq^{1-N})(1-a^{-1}z^{-1}q^{N-1})\cdots(-azq^{-1})(1-a^{-1}z^{-1}q)(az)_\infty$$

$$= (-az)^{N-1} q^{-N(N-1)/2} (a^{-1}z^{-1}q)_{N-1} (az)_\infty$$

$$= (-az)^{N-1} q^{-N(N-1)/2} \frac{(a^{-1}z^{-1}q)_\infty (az)_\infty}{(a^{-1}z^{-1}q^N)_\infty}$$

$$(a)_{1-N} = \frac{(a)_\infty}{(aq^{1-N})_\infty} = \frac{(a)_\infty (a^{-1}q^N)_\infty (-a)^{N-1} q^{N(N-1)/2}}{(a^{-1}q)_\infty (a)_\infty}$$

$$= \frac{(a^{-1}q^N)_\infty (-a)^{N-1} q^{N(N-1)/2}}{(a^{-1}q)_\infty}$$

$$(q^N)_{1-N} = \frac{(q^N)_\infty}{(q)_\infty}.$$

Hence,

$$ {}_1\psi_1\left(\begin{matrix} a \\ q^N \end{matrix}; q, z\right) = \frac{(a^{-1}q^N)_\infty}{(a^{-1}q)_\infty} \frac{(q)_\infty}{(q^N)_\infty} \frac{(a^{-1}z^{-1}q)_\infty (az)_\infty}{(a^{-1}z^{-1}q^N)_\infty (z)_\infty}$$

and (*) holds for $b = q^N$.

The result follows by the Lemma since both sides of (*) define an analytic function of b for $|b| < |az|$ that agrees for $b = q^N \to 0$ as $N \to \infty$. $\square$