<u>Definition</u> Let $k \geq 1$, $n \geq 0$.

Let $r_k(n)$ = number of ways of writing n as a sum of k squares

<u>ie</u>

$$r_k(n) = \{(x_1, x_2, \ldots, x_k) \in \mathbb{Z}^k : x_1^2 + x_2^2 + \cdots + x_k^2 = n\}$$

For example,

$r_2(1) = 4$ since

$1 = 0^2 + 1^2 = 1^2 + 0^2 = 0^2 + (-1)^2 = (-1)^2 + 0^2.$

<u>Theorem</u> (Jacobi) For $n \geq 1$,

$$r_2(n) = 4\left(\sum_{\substack{d \mid n \\ d \equiv 1 \pmod 4}} 1 - \sum_{\substack{d \mid n \\ d \equiv 3 \pmod 4}} 1\right).$$

<u>Example</u>: $r_2(1) = 4 = 4(1-0)$.

<u>Lemma</u>

$$\sum_{n=0}^{\infty} r_k(n) q^n = \left(\sum_{n=-\infty}^{\infty} q^{n^2}\right)^k$$

<u>Proof</u>: Let $|q|<1$.

$$\left(\sum_{n=-\infty}^{\infty} q^{n^2}\right)^k = \sum_{n_1 \in \mathbb{Z}} q^{n_1^2} \sum_{n_2 \in \mathbb{Z}} q^{n_2^2} \cdots \sum_{n_k \in \mathbb{Z}} q^{n_k^2}$$

$$= \sum_{(n_1, n_2, \ldots, n_k) \in \mathbb{Z}^k} q^{n_1^2 + \cdots + n_k^2}$$