$$= \sum_{n=0}^{\infty} \Big(\sum_{\substack{(n_1, n_2, \dots n_k) \in \mathbb{Z}^k \\ n_1^2 + \dots + n_k^2 = n}} 1 \Big) q^n$$

$$= \sum_{n=0}^{\infty} r_k(n) q^n. \qquad \square$$

<u>Lemma:</u> $$\sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = \prod_{n=1}^{\infty} \frac{(1-q^n)}{(1+q^n)} = \frac{(q;q)_\infty}{(-q;q)_\infty}$$

for $|q|<1$.

This was proved in Ch2 using JTP (see p.21 of Notes).

<u>Proof of Theorem:</u>

$${}_1\psi_1\left(\begin{matrix}-1\\-q\end{matrix}; q, i\sqrt{q}\right) = \sum_{n=-\infty}^{\infty} \frac{(i\sqrt{q})^n (-1;q)_n}{(-q;q)_n}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(-1;q)_n}{(-q;q)_n} i^n q^{n/2}$$

$$+ \sum_{n=1}^{\infty} \frac{(-1;q)_n}{(-q;q)_n} \left(\frac{-q}{-1(iq^{1/2})}\right)^n$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(-1;q)_n}{(-q;q)_n} (i^n + i^{-n}) q^{n/2}$$