~~Replacing q by~~

Hence
$$\left(\sum_{n=-\infty}^{\infty} (-1)^n q^{n^2}\right)^2 = 1 + 4\sum_{n=1}^{\infty} \frac{(-1)^n q^n}{1+q^{2n}}.$$

Replacing q by $-q$ we have

$$\sum_{n=0}^{\infty} r_2(n) q^n = \left(\sum_{n=-\infty}^{\infty} q^{n^2}\right)^2 = 1+4\sum_{n=1}^{\infty} \frac{q^n}{1+q^{2n}}$$

$$= 1+4\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} (-1)^m q^{n+2mn}$$

$$= 1+4\sum_{m=0}^{\infty} (-1)^m \sum_{k=1}^{\infty} q^{k(2m+1)}$$

$$= 1+4\sum_{m=0}^{\infty} \left(\sum_{k=1}^{\infty} q^{k(4m+1)} - \sum_{k=1}^{\infty} q^{k(4m+3)}\right)$$

$$= 1+4\left(\sum_{n=1}^{\infty}\left(\sum_{\substack{k(4m+1)=n \\ k\geq 1,\, 4m+1\geq 1}} 1\right) q^n - \cancel{4}\sum_{n=1}^{\infty}\left(\sum_{\substack{k(4m+3)=n \\ k\geq 1,\, m\geq 0}} 1\right) q^n\right)$$

$$= 1+4\left(\sum_{n=1}^{\infty}\left(\sum_{\substack{d\mid n \\ d\equiv 1 \pmod 4}} 1 - \sum_{\substack{d\mid n \\ d\equiv 3 \pmod 4}} 1\right) q^n\right)$$

Hence for $n\geq 1$,

$$r_2(n) = 4\left(\sum_{\substack{d\mid n \\ d\equiv 1 \pmod 4}} 1 - \sum_{\substack{d\mid n \\ d\equiv 3 \pmod 4}} 1\right) \quad \square$$