(ii) Hence, deduce from (*), Sylvesters result:
The number of partitions of n into distinct parts
= The number of self-conjugate partitions of n.

[10pts] (10) Prove

$$p(7n+5) \equiv 0 \pmod{7} \quad (\text{for } n \geq 0).$$

Hint: Write

$$\frac{1}{(q)_\infty} = \frac{\left((q)_\infty^3\right)^2}{(q)_\infty^7}$$

and use Jacobi's result that

$$(q)_\infty^3 = \sum_{n=0}^{\infty} (-1)^n (2n+1) q^{n(n+1)/2}.$$

[10pts] (11) Let $\{F_n\}_{n=0}^{\infty}$ be the Fibonacci sequence:

$F_0 = 1$
$F_1 = 1$
$F_2 = 2$
$F_3 = 3$
$F_4 = 5$ etc so that

$$F_n = F_{n-1} + F_{n-2} \quad \text{for } n \geq 2.$$

(a) Prove that (formally)

$$\mathcal{F}(q) = \sum_{n=0}^{\infty} F_n q^n = \frac{1}{1-q-q^2}$$

Hint: Show that $(1-q-q^2)\mathcal{F}(q) = 1$.