HW 5 – Due Thursday, June 9

(1) Prove that

$$\sum_{n=0}^{\infty} p(7n+5)q^n = 7\prod_{n=1}^{\infty}\frac{(1-q^{7n})^3}{(1-q^n)^4} + 49q\prod_{n=1}^{\infty}\frac{(1-q^{7n})^7}{(1-q^n)^8}$$

as follows:

[10pts] (a) Let $E(q) = (q)_\infty = \prod_{n=1}^{\infty}(1-q^n)$.
Prove that

$$E(q) = E(q^{49})\left[Q_0(q^7) - q\,Q_1(q^7) - q^2 + q^5 Q_5(q^7)\right]$$

where

$Q_0(q), Q_1(q), Q_5(q) \in \mathbb{Z}[[q]]$ and satisfy

(i) $Q_0 Q_1^2 - Q_0^2 + q^7 Q_5 = 0$

(ii) $Q_0 - Q_1^2 - q^7 Q_1 Q_5^2 = 0$

(iii) $Q_0^2 Q_5 - q^7 Q_5^2 - Q_1 = 0$

and (iv) $Q_0 Q_1 Q_5 = 1$.

[HINT: Prove (i) – (iv) by using

$$(E(q))^3 = \sum_{j\geq 0}(-1)^j(2j+1)q^{j(j+1)/2}\text{]}$$

[5pts] (b) Let $\alpha = -q^{-2}Q_0(q^7)$, $\beta = q^{-1}Q_1(q^7)$, & $\gamma = -q^3 Q_5(q^7)$
so that

$$X(q) := \frac{E(q)}{q^2 E(q^{49})} = -(\alpha+\beta+\gamma+1)$$