Show that

(i) $\alpha\beta^2 + \alpha^2 + \gamma = 0$

(ii) $\beta\gamma^2 + \beta^2 + \alpha = 0$

(iii) $\gamma\alpha^2 + \gamma^2 + \beta = 0$

and (iv) $\alpha\beta\gamma = 1$

[HINT: Use (i)–(iv) of (a)].

[10 pts] (c) Now let

$$y_1 = \alpha^3\beta, \quad y_2 = \beta^3\gamma, \quad \text{and} \quad y_3 = \gamma^3\alpha$$

Show that

(i) $\alpha^2\beta^3 = y_1 y_2 = -y_1 - 1$

(ii) $\beta^2\gamma^3 = y_2 y_3 = -y_2 - 1$

(iii) $\gamma^2\alpha^3 = y_3 y_1 = -y_3 - 1$

(iv) $y_1 y_2 y_3 = 1$

(v) $\alpha\beta^5 = y_1 - y_2 + 1$

(vi) $\beta\gamma^5 = y_2 - y_3 + 1$

(vii) $\gamma\alpha^5 = y_3 - y_1 + 1$

and hence find

α^7, β^7 and γ^7 in terms of y_1, y_2 and y_3.

[5 pts] (d) Let $\zeta = e^{2\pi i/7}$ so that $\zeta^7 = 1$ & $1+\zeta+\zeta^2+\zeta^3+\zeta^4+\zeta^5+\zeta^6 = 0$.

Show that

(i) $$\prod_{k=0}^{6} E(\zeta^k q) = \frac{E(q^7)^8}{E(q^{49})}$$

and (ii) $$\prod_{k=0}^{6} X(\zeta^k q) = \left(T(q)\right)^2 \quad \text{where} \quad T(q) = \left(\frac{E(q^7)}{q^7 E(q^{49})}\right)^4$$