

An identity that changed the course of history

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Introduction

In his first letter to Hardy, Ramanujan included the following three claims.

If

$$u = \frac{x}{1+} \frac{x^5}{1+} \frac{x^{10}}{1+} \frac{x^{15}}{1+} \frac{x^{20}}{1+} \dots$$

and

$$v = \frac{{}_5\sqrt{x}}{1+} \frac{x}{1+} \frac{x^2}{1+} \frac{x^3}{1+} \dots$$

then

$$v^5 = u \cdot \frac{1 - 2u + 4u^2 - 3u^3 + u^4}{1 + 3u + 4u^2 + 2u^3 + u^4}, \quad (1)$$

$$\frac{1}{1+} \frac{e^{-2\pi}}{1+} \frac{e^{-4\pi}}{1+} \frac{e^{-6\pi}}{1+} \dots = \left(\sqrt{\frac{5 + \sqrt{5}}{2}} - \frac{\sqrt{5} + 1}{2} \right) {}_5\sqrt{e^{2\pi}} \quad (2)$$

and

$$\frac{1}{1-} \frac{e^{-\pi}}{1-} \frac{e^{-2\pi}}{1-} \frac{e^{-3\pi}}{1-} \dots = \left(\sqrt{\frac{5 - \sqrt{5}}{2}} - \frac{\sqrt{5} - 1}{2} \right) {}_5\sqrt{e^{\pi}}, \quad (3)$$

while in his second letter he included the claim

$$\frac{1}{1+} \frac{e^{-2\pi\sqrt{5}}}{1+} \frac{e^{-4\pi\sqrt{5}}}{1+} \dots = \left[\frac{\sqrt{5}}{1 + {}_5\sqrt{\left\{ 5^{\frac{3}{4}} \left(\frac{\sqrt{5} - 1}{2} \right)^{\frac{5}{2}} - 1 \right\}}} - \frac{\sqrt{5} + 1}{2} \right] e^{2\pi\sqrt{5}}. \quad (4)$$

Twenty-seven years later, Hardy discusses the effect these identities had on him. “I should like you to begin by trying to reconstruct the immediate reactions of

an ordinary professional mathematician who receives a letter like this from an unknown Hindu clerk.” ... “The formulas (1), (2) and (4)” (Hardy got these a little confused here — (4) occurred only in the second letter) “are on a different level and obviously both difficult and deep. ... (1), (2) and (4) defeated me completely. I had never seen anything in the least like them before. **A single look at them is enough to show that they could only be written down by a mathematician of the highest class. They must be true because, if they were not true, no one would have had the imagination to invent them.**

Clearly, the three identities (1)–(3) had a great influence on Hardy’s opinion of Ramanujan, the ‘unknown Hindu clerk’.

Two days later, Bertrand Russell writes to Lady Ottoline Morrell “In Hall I found Hardy and Littlewood in a state of wild excitement, because they believe they have discovered a second Newton.”

Littlewood wrote “I can believe he’s at least a Jacobi.”

So I invite you to speculate on what might have (not) happened if Ramanujan had not included these three formulas in his first letter to Hardy. You might even speculate that Ramanujan may never have gone to England.

In any case, what I intend to do in this talk is to go some way to proving formula (1). I hope that you might agree with me that formula (1) might well be “an identity that changed the course of history”.