CHAPTER 1

Generalized Dedekind eta products

In [Bir75], Birch asked for a simple motivation and proof for Ramanujan's '40 identities'. By expanding on an idea of Rangachari [Ran88] we are able to predict all possible identities having various forms. In particular we discover that, for Hauptmoduls on $G \subseteq \Gamma$ of genus 0, each coset of G in Γ corresponds to a particular collection of identities. We can, of course, then proceed to prove all of these identities without too much trouble using the theory developed here. This would be tiresome if we were to carry it out for all of the new identities found, and in fact there are undoubtedly many more identities lurking among the Riemann surfaces of higher genus; we therefore prove one critical identity to give the general idea of the methods employed. While studying these functions, we also obtain a simple criterion which tells us that we only have to check the two points 0 and $i\infty$ to know when functions exist on $X_1(N)$. The functions we study are combinatorial objects, and as corollaries of the modular equations we get results about partitions. Modular equations also find applications in the study of singular moduli, elliptic curves with complex multiplication, and class numbers of orders in quadratic imaginary fields.

1.1 Some Background

Definition. Let

$$\eta_{\delta,g}(\tau) := e^{\pi i P_2(\frac{g}{\delta})\delta\tau} \prod_{\substack{m>0 \\ m\equiv g (mod\ \delta)}} (1-x^m) \prod_{\substack{m>0 \\ m\equiv -g (mod\ \delta)}} (1-x^m), \tag{1.1}$$

where $x = e^{2\pi i\tau}, \tau \in \mathcal{H}$, and where $P_2(t) = \{t\}^2 - \{t\} + \frac{1}{6}$ is the second periodic Bernoulli polynomial. Note that $\eta_{\delta,0} = x^{\frac{\delta}{12}} \prod_{n=1}^{\infty} (1-x^{\delta n})^2 = \eta(\delta\tau)^2$, so that these functions generalize the usual Dedekind eta function. The functions (1.1) are in fact a generalization of those appearing in the Rogers-Ramanujan identities, which in our notation are written as $\eta_{5,1}$ and $\eta_{5,2}$. The class of functions we study here is:

$$f(\tau) = \prod_{\substack{\delta|N \\ 0\leq g<\delta}} \eta_{\delta,g}^{r_{\delta,g}}, \tag{1.2}$$

where

$$r_{\delta,g} \in \begin{cases} \frac{1}{2}\mathbb{Z} & \text{if } g=0 \text{ or } g=\delta/2 \\ \mathbb{Z} & \text{otherwise.} \end{cases} \tag{1.3}$$

We proceed to find out how f transforms under an element $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ belonging to the modular subgroup $\Gamma_0(N)$; later we shall find that the subgroup

$$\Gamma_1(N) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N) \,\middle|\, a \equiv d \equiv 1 (\text{mod N}) \right\}$$

plays a crucial role in the study of the functions (1.2). We may assume without loss of generality that $(a, 6) = 1$, as in Newman's paper [New57]. Meyer ([Mey57] and [Mey60]) and others ([Die57,1],[Die57,2],[Sch74]) have worked out the transformation law for $\eta_{\delta,g}$ under any $A \in \Gamma$, the full modular group, and we quote the result below, following the notation of [Sch74]. Put

$$\eta_{g,h}^{(s)} = \alpha_0(h) e^{\pi i P_2(\frac{g}{\delta})\tau} \prod_{\substack{m>0 \\ m\equiv g (mod\ \delta)}} (1 - \zeta_\delta^h e^{\frac{2\pi i\tau}{\delta} m}) \prod_{\substack{m>0 \\ m\equiv -g (mod\ \delta)}} (1 - \zeta_\delta^{-h} e^{\frac{2\pi i\tau}{\delta} m}),$$

where ζ_δ is a primitive δ 'th root of unity,

$$P_1(x) := ((x)) := \begin{cases} x - [x] - \frac{1}{2} & \text{if } x \notin \mathbf{Z} \\ 0 & \text{if } x \in \mathbf{Z} \end{cases} \tag{1.4}$$

is the first periodic Bernoulli polynomial, and

$$\alpha_0(h) = \begin{cases} (1 - \zeta_\delta^{-h}) e^{\pi i P_1(\frac{h}{\delta})} & \text{if } g \equiv 0 \text{ and } h \not\equiv 0 \pmod{\delta} \\ 1 & \text{otherwise.} \end{cases} \tag{1.5}$$

We can assume without loss of generality that $0 \leq g < \delta$.

1.2 The transformation law

Our functions are related to those of Schoeneberg as follows:

$$\eta_{\delta,g}(\tau) = \eta_{g,0}^{(s)}(\delta\tau). \tag{1.6}$$

Thus to get $\eta_{\delta,g}(A\tau)$ for $A \in \Gamma_0(\delta)$, we need $\eta_{g,0}^{(s)}(\delta A\tau) = \eta_{g,0}^{(s)}(A_1\delta\tau)$, where $A_1 = \left(\begin{smallmatrix} a & \delta b \\ c/\delta & d \end{smallmatrix}\right)$. If $g \neq 0$, then by [Sch74] we have

$$\eta_{g,0}^{(s)}(A_1\delta\tau) = e^{\pi i \mu_{\delta,g}} \eta_{g',h'}^{(s)}(\delta\tau), \tag{1.7}$$

where

$$\begin{pmatrix} g' \\ h' \end{pmatrix} = A_1^t \begin{pmatrix} g \\ 0 \end{pmatrix}, \tag{1.8}$$

$$\mu_{\delta,g} := \frac{\delta a}{c} P_2(\frac{g}{\delta}) + \frac{\delta d}{c} P_2(\frac{ag}{\delta}) - 2s(a, \frac{c}{\delta}; 0, \frac{g}{\delta}), \tag{1.9}$$

and $s(h,k;x,y)$ is the Meyer Sum, a generalized Dedekind sum, defined by

$$s(h,k;x,y) = \sum_{\mu mod k} \left(\left(h\left(\frac{\mu+y}{k}\right) + x \right) \right) \left(\left(\frac{\mu+y}{k} \right) \right). \tag{1.10}$$

Equation (1.8) translates into

$$\begin{pmatrix} a & \frac{c}{\delta} \\ \delta b & d \end{pmatrix} \begin{pmatrix} g \\ 0 \end{pmatrix} = \begin{pmatrix} g' \\ h' \end{pmatrix},$$

so that $g' = ag$,and $h' = \delta bg$. Note here that if $a \equiv 1 \pmod{\delta}$, then $g' \equiv g$ and $h' \equiv 0 \bmod \delta$; that is, A_1^t takes $\binom{g}{0}$ to itself mod δ and thus takes $\eta_{\delta,g}$ to itself. This observation suggests the use of $\Gamma_1(N)$. ¿From (1.8) and the above remarks, we have

$$\eta_{\delta,g}(A\tau) = \eta_{g,0}^{(s)}(\delta A\tau) = \eta_{g,0}^{(s)}(A_1\delta\tau) = e^{\pi i \mu_{\delta,g}} \eta_{ag,0}^{(s)}(\delta\tau) = e^{\pi i \mu_{\delta,g}} \eta_{\delta,ag}(\tau).$$

Hence

Theorem 1.1

$$f(A\tau) = \prod_{\substack{\delta|N \\ g}} \eta_{\delta,g}(A\tau)^{r_{\delta,g}} = \prod_{\substack{\delta|N \\ g}} \eta_{\delta,ag}(\tau)^{r_{\delta,g}} e^{\pi i \mu_{\delta,g} r_{\delta,g}}.$$

We need to find $\mu_{\delta,g}$ mod 2 in order to find necessary and sufficient conditions for $f(\tau) \in S_k\big(\Gamma_1(N)\big)$. By the reciprocity law for the Meyer sums [RG72], we have

$$-s(a, \frac{c}{\delta}; 0, \frac{g}{\delta}) = s(\frac{c}{\delta}, a; \frac{g}{\delta}, 0) - \frac{a\delta}{2c} P_2(\frac{g}{\delta}) - \frac{\delta}{2ac} P_2(\frac{ag}{\delta}) - \frac{c}{2\delta a} P_2(0).$$

Now suppose that $A \in \Gamma_1(N)$, so that $a \equiv 1 \pmod N$. Then $a \equiv 1 \pmod \delta$ and hence $P_2(\frac{ag}{\delta}) = P_2(\frac{g}{\delta})$, as P_2 is periodic mod 1. Thus

$$\begin{aligned}
\mu_{\delta,g} &= \frac{\delta a}{c}P_2(\frac{g}{\delta}) + \frac{\delta d}{c}P_2(\frac{ag}{\delta}) + 2s(\frac{c}{\delta}, a; \frac{g}{\delta}, 0) - \frac{a\delta}{c}P_2(\frac{g}{\delta}) - \frac{\delta}{ac}P_2(\frac{g}{\delta}) - \frac{c}{\delta a}P_2(0) \\
&= \frac{\delta ad}{ac}P_2(\frac{ag}{\delta}) + 2s(\frac{c}{\delta}, a; \frac{g}{\delta}, 0) - \frac{\delta}{ac}P_2(\frac{g}{\delta}) - \frac{c}{6\delta a} \\
&= 2s(\frac{c}{\delta}, a; \frac{g}{\delta}, 0) + \frac{\delta b}{a}P_2(\frac{g}{\delta}) - \frac{c}{6\delta a},
\end{aligned}$$

where we used $ad - 1 = bc$ in the last step. We now expand and simplify the Meyer sum:

$$s(\frac{c}{\delta}, a; \frac{g}{\delta}, 0) = \sum_{\nu=1}^{a-1}\left(\left(\frac{\nu}{a}\right)\right)\left(\left(\frac{c}{\delta}\frac{\nu}{a} + \frac{g}{\delta}\right)\right).$$

The sum is actually from $\nu = 0$ to $\nu = a - 1$, but $\left(\left(\frac{\nu}{a}\right)\right) = 0$ when $\nu = 0$. Hence

$$\mu_{\delta,g} = 2\sum_{\nu=1}^{a-1}\left(\left(\frac{\nu}{a}\right)\right)\left(\left(\frac{c}{\delta}\frac{\nu}{a} + \frac{g}{\delta}\right)\right) - \frac{c}{6\delta a} + \frac{\delta b}{a}P_2(\frac{g}{\delta})$$

Note: $(a, 6) = 1$ means $a \equiv 1$ or $5 \pmod 6$, which implies that $a^2 \equiv 1 \pmod{24}$. By [Sch74], if $\zeta := e^{\pi i \mu_{\delta,g}}$, then $\zeta^{N_2} = 1$, where $N_2 := \frac{12N}{(6,N)}$.

Lemma 1.1

$$\zeta^{a^2} = \zeta$$

Proof: It suffices to prove that $a^2 \equiv 1 \pmod{N_2}$.

Case 1: $(N, 6) = 1$. $a^2 \equiv 1 \pmod N$ and $a^2 \equiv 1 \pmod{24} \Rightarrow a^2 \equiv 1 \pmod{12N}$.

Case 2: $(N, 6) = 2$. $a \equiv 1 \pmod N \Rightarrow a = 1 + jN \Rightarrow a^2 = 1 + 2jN + j^2N^2 \equiv 1 \pmod{2N}$, and $a^2 \equiv 1 \pmod 3$, so $a^2 \equiv 1 \pmod{6N}$.

Case 3: $(N,6)=3$. Now $N_2=4N$. $a^2 \equiv 1 \pmod 4$, and $a^2 \equiv 1 \pmod N$ $\Rightarrow a^2 \equiv 1 \pmod{4N}$.

Case 4: $(N,6)=6$. Now $N_2=2N$. Again, $a=1+jN \Rightarrow a^2 = 1+2jN+j^2N^2 \equiv 1 \pmod{2N}$. $\square$

It follows from this lemma that

$$\mu_{\delta,g} \equiv 2a^2 \sum_{\nu=1}^{a-1} \Big(\frac{\nu}{a} - \frac{1}{2}\Big)\left(\left(\frac{c}{\delta}\frac{\nu}{a} + \frac{g}{\delta}\right)\right) - \frac{ac}{6\delta} + ab\delta P_2\Big(\frac{g}{\delta}\Big) \pmod 2.$$

Now $\delta \mid N$ and $N \mid c \Longrightarrow \delta \mid c$, and $(a,c)=1 \Longrightarrow (a,\frac{c}{\delta})=1$, so that $\frac{c}{\delta}\nu$ runs through a complete residue system mod a as ν does. Hence

$$\mu_{\delta,g} \equiv ab\delta P_2\Big(\frac{g}{\delta}\Big) - \frac{ac}{6\delta} + 2a\sum_{\nu=1}^{a-1} \nu\Big(\Big(\frac{c\nu}{\delta a} + \frac{g}{\delta}\Big)\Big) - \sum_{\nu=1}^{a-1}\Big(\Big(\frac{\nu}{a} + \frac{g}{\delta}\Big)\Big) \pmod 2.$$

· But this last term is zero, since

$$\begin{aligned}\sum_{\nu=1}^{a-1}\Big(\Big(\frac{\nu}{a}+\frac{g}{\delta}\Big)\Big) &= \sum_{\nu=1}^{a-1}\Big(\frac{\nu}{a}+\frac{g}{\delta}\Big) - \sum_{\nu=1}^{a-1}\Big[\frac{\nu}{a}+\frac{g}{\delta}\Big] - \sum_{\nu=1}^{a-1}\frac{1}{2} \\ &= \frac{a-1}{2} + \Big(\frac{g}{\delta}-\frac{1}{2}\Big)(a-1) - \Big(\Big[a\frac{g}{\delta}\Big] - \Big[\frac{g}{\delta}\Big]\Big),\end{aligned}$$

where we have used the identity $\sum_{\nu=0}^{a-1}\Big[\frac{\nu}{a}+x\Big] = [ax]$ in the last step. Now $\Big[\frac{g}{\delta}\Big]=0$ since $0 \le g < \delta$, and $\Big[a\frac{g}{\delta}\Big] = \frac{g}{\delta}(a-1)$ since $a \equiv 1(\delta)$. Hence

$$\sum_{\nu=1}^{a-1}\Big(\Big(\frac{\nu}{a}+\frac{g}{\delta}\Big)\Big) = \frac{g}{\delta}(a-1) - \frac{g}{\delta}(a-1) = 0.$$

Thus

$$\mu_{\delta,g} \equiv ab\delta P_2\Big(\frac{g}{\delta}\Big) - \frac{ac}{6\delta} + 2a\sum_{\nu=1}^{a-1}\nu\left(\Big(\frac{c\nu}{\delta a}+\frac{g}{\delta}\Big) - \Big[\frac{c\nu}{\delta a}+\frac{g}{\delta}\Big] - \frac{1}{2}\right) \pmod 2$$

Note that $2a\sum \frac{c\nu^2}{\delta a}$ and $2a\sum\left[\frac{c\nu}{\delta a}+\frac{g}{\delta}\right]$ are even integers, so that

$$\begin{aligned} \mu_{\delta,g} &\equiv ab\delta P_2(\frac{g}{\delta}) - \frac{ac}{6\delta} + a^2(a-1)\frac{g}{\delta} - \frac{1}{2}a^2(a-1) \pmod 2 \quad (1.11) \\ &\equiv ab\delta P_2(\frac{g}{\delta}) - \frac{ac}{6\delta} + (a-1)\frac{g}{\delta} - \frac{1}{2}(a-1) \pmod 2, \quad (1.12) \end{aligned}$$

since $a^2 \equiv 1 \pmod 2$.

1.3 A useful criterion

Now recall that (with the assumption that $a \equiv 1$ (mod N)),

$$\begin{aligned} f(A\tau) &= \prod_{\substack{\delta|N \\ g}} \eta_{\delta,g}(\tau)^{r_{\delta,g}} e^{\pi i \mu_{\delta,g} r_{\delta,g}} \quad (1.13) \\ &= f(\tau) e^{\pi i \sum \mu_{\delta,g} r_{\delta,g}}. \quad (1.14) \end{aligned}$$

Setting the coefficients of ab and of ac in $\sum \mu_{\delta,g} r_{\delta,g}$ congruent to zero mod 2 we get, by (1.12):

$$\sum_{\substack{\delta|N \\ g}} \delta P_2(\frac{g}{\delta}) r_{\delta,g} \equiv 0 \pmod 2 \quad (1.15)$$

$$\sum_{\substack{\delta|N \\ g}} \delta' P_2(0) r_{\delta,g} \equiv 0 \pmod 2. \quad (1.16)$$

We now claim that these 2 congruences make $\sum \mu_{\delta,g} r_{\delta,g} \equiv 0 \pmod 2$. By (1.12), this amounts to proving

Lemma 1.2 *If (1.15) and (1.16) hold, then*

$$\sum_{\substack{\delta|N \\ g}} (a-1)\frac{g}{\delta} r_{\delta,g} \equiv 0 \equiv \sum_{\substack{\delta|N \\ g}} \frac{(a-1)}{2} r_{\delta,g} \quad (mod\ 2).$$

Before proceeding with the proof, we shall make some useful remarks and transformations. We can clearly expand $\eta_{\delta,g}$ into

$$\eta_{\delta,g} = \eta_{N,g}\eta_{N,g+\delta} \ \cdots \ \eta_{N,g+(\delta'-1)\delta}. \tag{1.17}$$

The only detail left to check is that the multiplier systems on both sides agree. This amounts to proving

Lemma 1.3

$$\delta P_2\left(\frac{g}{\delta}\right) = N \sum_{\substack{g'=g+m\delta \\ m=0,1,\dots,\delta'-1}} P_2\left(\frac{g'}{N}\right).$$

Proof

$$\begin{aligned}
N \sum_{\substack{g'=g+m\delta \\ m=0,1,\dots,\delta'-1}} P_2\left(\frac{g'}{N}\right) &= N \sum_{\substack{g'=g+m\delta \\ m=0,1,\dots,\delta'-1}} \left[\left(\frac{g'}{N}\right)^2 - \frac{g'}{N} + \frac{1}{6}\right] \\
&= N \sum_{m=0}^{\delta'-1} \left[\left(\frac{g+m\delta}{N}\right)^2 - \frac{g+m\delta}{N} + \frac{1}{6}\right] \\
&= \frac{1}{N}\sum_{m=0}^{\delta'-1} [g^2 + 2\delta gm + \delta^2 m^2] - \sum_{m=0}^{\delta'-1} (g+m\delta) + \frac{N}{6}\delta' \\
&= \frac{g^2}{N}\delta' + \frac{\delta g}{N}(\delta'-1)\delta' + \frac{\delta^2}{6N}(\delta'-1)\delta'(2\delta'-1) - g\delta' - \frac{\delta}{2}(\delta'-1)\delta' + \frac{N}{6}\delta' \\
&= \frac{g^2}{\delta} + (\delta'-1)\left(g + \frac{\delta}{6}(2\delta'-1) - \frac{N}{2}\right) - g\delta' + \frac{\delta' N}{6} \\
&= \delta\left(\frac{g^2}{\delta^2} - \frac{g}{\delta} + \frac{1}{6}\right) \\
&= \delta P_2\left(\frac{g}{\delta}\right).
\end{aligned}$$

The uniformizing process of replacing every δ dividing N by N itself is useful in proving these lemmas and leads to the following formulation:

$$f(\tau) = \prod_{\substack{\delta|N \\ g}} \eta_{\delta,g}(\tau)^{r_{\delta,g}} = \prod_{g'} \eta_{N,g'}^{t_{g'}},$$

where

$$t_{g'} = \sum_{\substack{g'=g+m\delta \\ m=0,1,\ldots,\delta'-1}} r_{\delta,g'}$$

from (1.17). Sometimes we shall abuse notation and replace the delimeters on the summation sign by the single letter g'.

Lemma 1.4

$$\sum_{\substack{\delta|N \\ g}} \delta P_2\left(\frac{g}{\delta}\right) r_{\delta,g} = \sum_{g'} N P_2\left(\frac{g'}{N}\right) t_{g'},$$

where $t_{g'} = \sum_{g'} r_{\delta,g}$.

Proof.

$$\begin{aligned}
\sum_{\substack{\delta|N \\ g}} \delta P_2\left(\frac{g}{\delta}\right) r_{\delta,g} &= \sum_{\substack{\delta|N \\ g}} \sum_{g'} N P_2\left(\frac{g'}{N}\right) r_{\delta,g} \\
&= \sum_{g'} \left[N P_2\left(\frac{g'}{N}\right) \sum_{g'} r_{\delta,g} \right] \\
&= \sum_{g'} N P_2\left(\frac{g'}{N}\right) t_{g'},
\end{aligned}$$

where lemma 1.3 us used in the first step. □

Lemma 1.5

$$\sum_{\substack{\delta|N \\ g}} \delta' P_2(0) r_{\delta,g} = \sum_{g'} P_2(0) t_{g'}$$

Proof.

$$\begin{aligned}
\sum_{g'} t_{g'} &= \sum_{g'} \sum_{\substack{\delta|N,g \\ \ni g'=g+m\delta}} r_{\delta,g} \\
&= \sum_{\substack{\delta|N \\ g}} \sum_{\substack{\delta|N,g,m \\ \ni g'=g+m\delta}} r_{\delta,g} \\
&= \sum_{\substack{\delta|N \\ g}} \delta' r_{\delta,g}.
\end{aligned}$$

□

We remark that $\sum \frac{g}{\delta} r_{\delta,g} = \sum \frac{g'}{N} t_{g'}$ and $\sum r_{\delta,g} = \sum t_{g'}$. Their proofs parallel the proof of lemma 1.5 and are therefore omitted. Lemma 1.2 then becomes

Lemma 1.6 *If (1.15) and (1.16) hold, then*

$$\frac{(a-1)}{N}\sum_{g'} g' t_{g'} \equiv 0 \equiv \sum_{g'} \frac{(a-1)}{2} t_{g'} \quad (mod 2).$$

Proof: Lemma 1.5 and the congruence (1.16) give

$$\frac{1}{6}\sum_{g'} t_{g'} \equiv 0 \pmod 2 \tag{1.18}$$

Also, $(a,6) = 1$ implies that $a - 1/2$ is an integer, so that the second congruence of the lemma is proved.

For the first congruence, lemma 1.4 and the congruence (1.15) give

$$0 \equiv \sum_{g'} N P_2(\frac{g'}{N}) t_{g'} \tag{1.19}$$

$$\equiv \sum_{g'} N\Big(\frac{g'^2}{N^2} - \frac{g'}{N} + \frac{1}{6}\Big) t_{g'} \tag{1.20}$$

$$\equiv \sum_{g'} \Big(\frac{g'^2}{N} - g'\Big) t_{g'} + \frac{N}{6}\sum_{g'} t_{g'} \tag{1.21}$$

$$\equiv \sum_{g'} \Big(\frac{g'^2}{N} - g'\Big) t_{g'} \pmod 2, \tag{1.22}$$

where $\frac{N}{6}\sum_{g'} t_{g'} \equiv 0$ by (1.18). Hence

$$\sum \Big(g'^2 - g'N\Big) t_{g'} \equiv 0 \pmod 2. \tag{1.23}$$

Notice that all of the $t_{g'}$ are integers, because the assumption that the weight k is an integer means $k = \frac{1}{2}\sum_{\delta|N} r_{\delta,0} = \frac{1}{2} t_0 \in \mathbf{Z}$, and (1.18) implies $\sum t_{g'} \in \mathbf{Z}$. Thus

$t_{\frac{N}{2}} \in \mathbf{Z}$ if N is even . (t_0 and $t_{\frac{N}{2}}$ were the only two exponents in "doubt" of being integers). Thus $\sum_{g'} g' t_{g'} \in \mathbf{Z}$.

Case 1: N odd. $a \equiv 1 \pmod 2$ and $a \equiv 1$ (mod N) means $\frac{a-1}{N}$ is an even integer. Hence $\frac{a-1}{N} \sum_{g'} g' t_{g'} \equiv 0 \pmod 2$.

Case 2: N even. $\sum_{g'} g' t_{g'} \equiv \sum_{g'} g'^2 t_{g'} \equiv N \sum_{g'} g' t_{g'}$, by (1.22), and the last term is an even integer, since N is even. The first congruence follows from $i^2 \equiv i$ (mod 2) for any $i \in \mathbf{Z}$, and from the fact that each $t_{g'}$ is an integer. □

Theorem 1.2 *If (1.15) and (1.16) hold, then f is on $\Gamma_1(N)$.*

Proof: Lemma 1.2 implies $\sum \mu_{\delta,g} r_{\delta,g} \equiv 0 \pmod 2$, which implies that the root of unity in (1.14) is 1. □

Remark: [New57] obtained a similar result for the usual η-products on $\Gamma_0(N)$.

1.4 The orders at the cusps

A general cusp for $\Gamma_1(N)$ is given by $\kappa = \frac{\lambda}{\mu\epsilon}$, where $\epsilon \mid N$, and $(\lambda, N) = (\lambda, \mu) = (\mu, N) = 1$. Let $A_0 = \left(\begin{smallmatrix} \lambda & b_0 \\ \mu\epsilon & d_0 \end{smallmatrix}\right) \in \Gamma$, so that $\lambda d_0 - \mu\epsilon b_0 = 1$. Then A takes $i\infty$ to $\frac{\lambda}{\mu\epsilon} = \kappa$. Because $\eta_{\delta,g}(A\tau) = \eta_{g,0}^{(s)}(\delta A\tau)$, we need to simplify $\delta A\tau$.

$$\begin{aligned}\delta A\tau = \begin{pmatrix} \delta\lambda & \delta b_0 \\ \mu\epsilon & d_0 \end{pmatrix}\tau &= \frac{\delta\lambda\tau + \delta b_0}{\mu\epsilon\tau + d_0} \\ &= \frac{\delta\lambda\tau + \delta b_0}{\mu\epsilon\tau + d\frac{\delta\lambda}{D} - b\frac{\mu\epsilon}{D}},\end{aligned}$$

where $D = (\delta\lambda, \mu\epsilon) = (\delta, \epsilon)$, since $\delta \mid N, \epsilon \mid N$, and $(\lambda, N) = 1 = (\mu, N)$. Let $\delta_0 := \frac{\delta\lambda}{D}$, and $\epsilon_0 := \frac{\mu\epsilon}{D}$. ¿From above,

$$\begin{aligned}\delta A\tau &= \frac{D\frac{\delta\lambda}{D}\tau + \delta b_0}{D\frac{\mu\epsilon}{D}\tau + d\frac{\delta\lambda}{D} - b\frac{\mu\epsilon}{D}} \\ &= \frac{\delta_0(D\tau - b) + \delta_0 b + \delta b_0}{\epsilon_0(D\tau - b) + d\delta_0} \\ &= \frac{\delta_0\frac{D\tau - b}{\delta/D} + \frac{\delta_0 b + \delta b_0}{\delta/D}}{\epsilon_0\frac{D\tau - b}{\delta/D} + \frac{d\delta_0}{\delta/D}} \\ &= \frac{\delta_0\frac{D\tau - b}{\delta/D} + \lambda b + b_0 D}{\epsilon_0\frac{D\tau - b}{\delta/D} + d\lambda} \\ &= \begin{pmatrix} \delta_0 & \lambda b + b_0 D \\ \epsilon_0 & d\lambda \end{pmatrix}\left(\frac{D\tau - b}{\delta/D}\right) \\ &:= A_0\left(\frac{D\tau - b}{\delta/D}\right),\end{aligned}$$

and we note that now $A_0 \in \Gamma$ because $\delta_0 d\lambda - \epsilon_0\lambda b - \epsilon_0 b_0 D = \lambda(d\delta_0 - b\epsilon_0) - \epsilon_0 b_0 D = \lambda d_0 - b_0\epsilon_0 D = \lambda d_0 - b_0\mu\epsilon = 1$. Therefore

$$\begin{aligned}
\eta_{\delta,g}(A\tau) &= \eta_{g,0}^{(s)}(\delta A\tau) \\
&= \eta_{g,0}^{(s)}\left(A_0\left(\frac{D\tau - b}{\delta/D}\right)\right) \\
&= e^{\pi i \mu_{\delta,g}}\eta_{\delta_0 g,h'}^{(s)}\left(\frac{D\tau - b}{\delta/D}\right),
\end{aligned}$$

since $A_0^t\binom{g}{0} = \binom{\delta_0 g}{h'}$. Consequently,

$$\begin{aligned}
\eta_{\delta,g}(A\tau) &= e^{\pi i \mu_{\delta,g}} e^{2\pi i \frac{1}{2} P_2(\frac{\delta_0 g}{\delta})\left(\frac{D\tau - b}{\epsilon/D}\right)} + \text{higher order terms} \\
&= \zeta x^{\frac{1}{2}P_2(\frac{\delta_0 g}{\delta})\frac{D^2}{\delta}},
\end{aligned}$$

where ζ is a root of unity. Hence

Theorem 1.3 *The order of $f(A\tau)$ in the uniformizing variable $x^{\frac{\epsilon}{N}}$ is*

$$\frac{N}{\epsilon}\frac{1}{2}\sum_{\substack{\delta|N \\ g}} \frac{D^2}{\delta} P_2\left(\frac{\delta_0 g}{\delta}\right) r_{\delta,g}$$

$$= \frac{N}{2}\sum_{\substack{\delta|N \\ g}} \frac{(\delta,\epsilon)^2}{\delta\epsilon} P_2\left(\frac{\lambda g}{(\delta,\epsilon)}\right) r_{\delta,g}.$$

□

Corollary 1.1 *If the order of f at 0 and at $i\infty$ is an integer, then f is on $\Gamma_1(N)$.*

Proof: For the cusp $\kappa = 0 \sim 1$, the above formula reduces to (1.15), and for the cusp $\kappa = i\infty \sim \frac{1}{N}$, the above formula reduces to (1.16). By theorem 1.2, the result follows. □

1.5 New Identities

Throughout, we use the notation $\eta_m := \eta(m\tau)$.

N = 5: Here $G(n) := \eta_{5,1}^{-1}(n\tau)$ and $H(n) := \eta_{5,2}^{-1}(n\tau)$.

$$\frac{G(2)G(63)+H(2)H(63)}{G(126)G(1)-H(126)H(1)} = \frac{\eta_3\eta_7\eta_{18}\eta_{42}}{\eta_6\eta_{14}\eta_9\eta_{21}}. \tag{1.24}$$

$$G^2(1)H(2) - H^2(1)G(2) = 2H(1)H^2(2)\frac{\eta_{10}^2}{\eta_5^2}. \tag{1.25}$$

$$G^2(1)H(2) + H^2(1)G(2) = 2G(1)G^2(2)\frac{\eta_{10}^2}{\eta_5^2}. \tag{1.26}$$

$$G^3(1)H(3) - G(3)H^3(1) = 3\frac{\eta_{15}^3}{\eta_1\eta_3\eta_5}. \tag{1.27}$$

N = 8: Here $G(n) := \eta_{8,1}^{-1}(n\tau)$ and $H(n) := \eta_{8,3}^{-1}(n\tau)$.

$$G(7)H(1) - G(1)H(7) = \frac{\eta_4\eta_{28}}{\eta_8\eta_{56}} \tag{1.28}$$

$$G(3)H(1) + G(1)H(3) = \frac{\eta_2\eta_4^2\eta_6^2}{\eta_1\eta_3\eta_8^2\eta_{12}} \tag{1.29}$$

$$G(3)H(1) - G(1)H(3) = \frac{\eta_3\eta_4^2}{\eta_1\eta_8\eta_{24}} \tag{1.30}$$

$$G^2(1) + H^2(1) = \frac{\eta_2^6}{\eta_1^3\eta_4\eta_8^2} \tag{1.31}$$

$$G^2(1) - H^2(1) = \frac{\eta_4^6}{\eta_1\eta_2\eta_8^4} \tag{1.32}$$

$$\left(\frac{\eta_{8,3}}{\eta_{8,1}}\right)^2 + \left(\frac{\eta_{8,1}}{\eta_{8,3}}\right)^2 = 6 + \frac{\eta_1^4\eta_4^2}{\eta_2^2\eta_8^4} \tag{1.33}$$

N = 12: Here $G(n) := \eta_{12,1}^{-1}(n\tau)$ and $H(n) := \eta_{12,5}^{-1}(n\tau)$.

$$G^2(1) - H^2(1) = \frac{\eta_2^3\eta_6^3}{\eta_1^2\eta_{12}^4} \tag{1.34}$$

$$G^2(1) + H^2(1) = \frac{\eta_2\eta_3^4\eta_4}{\eta_1^2\eta_6\eta_{12}^3} \tag{1.35}$$

$$G(3)H(1) + G(1)H(3) = \frac{\eta_4\eta_6^5\eta_9^2}{\eta_2\eta_3^2\eta_{12}^3\eta_{18}^2} \tag{1.36}$$

$$G(3)H(1) - G(1)H(3) = \frac{\eta_2\eta_{18}}{\eta_{12}\eta_{36}} \tag{1.37}$$

N = 13: Here $G(n) := \eta_{13,1}^{-1}\eta_{13,3}^{-1}\eta_{13,4}^{-1}(n\tau)$ and $H(n) := \eta_{13,2}^{-1}\eta_{13,5}^{-1}\eta_{13,6}^{-1}(n\tau)$.

$$G(3)H(1) - G(1)H(3) = 1 \tag{1.38}$$

1.6 Proving the Idntities

As we discussed in the beginning of the chapter, we will confine our attention to one identity, namely (1.38). Using corollary 1.1, we immediately find that $G(3)H(1)$ and $G(1)H(3)$ are functions on $\Gamma_R(39)$, and their orders at the three cusps $\frac{1}{13}, \frac{2}{13}$ and 1 of Γ_R are identical. Moreover, we only have to check the first few coefficients to realize that the left hand side is holomorphic on a compact Riemann surface, hence constant by Liouville's theorem. The constant is easily checked to be 1.

All of the other identities are proved in a similar fashion: dividing both sides by the right hand η-product, and finding the Riemann surface on which all of the terms live. We then check the cusps using theorem 1.3, and the poles will all cancel, leaving us with a holomorphic function on a compact Riemann surface, which must therefore be constant.

1.7 Discovering New Identities

In this section, we expand on an idea of Rangachari [Ran88] to discover that each coset of $\Gamma_1(N)$ in $\Gamma_0(N)$ corresponds to a particular type of identity, and predicts all such possible identities. The theory developed predicts that for genus

0 Riemann surfaces various combinations of generalized Dedekind η-products are non-zero. We then check to see if those combinations are η-products. We find that in many cases they are not, a curious fact in itself.

N = 5: By the formula for the orders at the cusps (theorem 1.3) we find that $\eta_{5,1}/\eta_{5,2}$ has a simple pole at one cusp and no other poles at any of the other cusps of $\Gamma(5)$. That is, $\eta_{5,1}/\eta_{5,2}$ is a Hauptmodul for $X(5)$, a Riemann surface of genus 0. We furthermore find that $A = \left(\begin{smallmatrix} 1 & 1 \\ 5 & 6 \end{smallmatrix}\right)$ has order 2 and $B = \left(\begin{smallmatrix} -3 & -5 \\ 5 & 8 \end{smallmatrix}\right)$ has order 5 in $\Gamma_0(5)/\Gamma(5)$, while their commutator $ABA^{-1}B^{-1} \notin \Gamma(5)$. This means $\Gamma_0(5)/\Gamma(5)$ is a non-abelian group of order 10, so must be D_5, the Dihedral group. Let $G(n\tau) := \eta_{5,1}(n\tau)$ and $H(n\tau) := \eta_{5,2}(n\tau)$. Applying the transformation rule, we get

$$\frac{G}{H}(B\tau) = -\frac{H}{G}(\tau) \tag{1.39}$$

$$\frac{G}{H}(A\tau) = e^{\frac{2\pi i}{5}}\frac{G}{H}(\tau). \tag{1.40}$$

The second equation doesn't give rise to identities, but the first one does. Any identity of the form

$$G(n\tau)G(m\tau) + H(n\tau)H(m\tau) = \eta - \text{product} \tag{1.41}$$

must satisfy

$$\begin{aligned} G(n\tau)G(m\tau) + H(n\tau)H(m\tau) &\neq 0 \\ \Leftrightarrow \frac{G}{H}(n\tau) &\neq -\frac{H}{G}(m\tau) \\ \Leftrightarrow \frac{G}{H}(n\tau) &\neq \frac{G}{H}(Bm\tau), \end{aligned}$$

where equation (1.39) was used in the last step. Since $\frac{G}{H}$ is a Hauptmodul for $\Gamma(5)$, it takes each value in $\mathcal{H}$ exactly once. Thus the last inequality holds

$$\begin{aligned} \Leftrightarrow n\tau &\not\sim B m\tau \text{ under } \Gamma(5) \\ \Leftrightarrow n\tau &\not\sim m\tau \text{ under the coset } B\Gamma(5) \\ \Leftrightarrow n\tau &\neq \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) m\tau \text{ with } \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in B\Gamma(5) \\ \Leftrightarrow cmn\tau^2 + (dn - am)\tau - b &\neq 0 \text{ with } \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in B\Gamma(5) \\ \Leftrightarrow (dn + am)^2 - 4mn &\geq 0. \end{aligned}$$

This condition on the discriminant of the quadratic in τ gives all of the possible identities of the form (1.41), and Ramanujan actually found them all [Bir75]. We have, however, discovered 4 new identities involving $\eta_{5,1}$ and $\eta_{5,2}$, the first of which Ramanujan must have overlooked, since he had a few in an identical form, and the next 3 of which we cannot yet motivate. The first is found by forcing the zeroes of the numerator to equal the zeroes of the denominator, thereby getting a non-zero quotient on $\mathcal{H}$.

N = 8: Again using theorem 1.3, we find that $\eta_{8,1}/\eta_{8,3}$ has a simple pole at one cusp and no other poles at any of the other cusps of $\Gamma_1(8) \cap \Gamma^0(8)$, which we will denote by $\Gamma_1(8,2)$. That is, $\eta_{8,1}/\eta_{8,3}$ is a Hauptmodul for $X_1(8,2)$, a Riemann surface of genus 0. We furthermore find that $[\Gamma_0(8) : \Gamma_1(8,2)] = 4$, and $\Gamma_0(8) \big/ \Gamma_1(8,2)$ is the Klein group of order 4. It has coset representatives $I, A =$

$\left(\begin{smallmatrix} 3 & 1 \\ 8 & 3 \end{smallmatrix}\right)$, $B = \left(\begin{smallmatrix} 3 & 4 \\ 8 & 11 \end{smallmatrix}\right)$, and $C = AB$. Let $G(n\tau) := \eta_{8,1}(n\tau)$ and $H(n\tau) := \eta_{8,3}(n\tau)$. By the transformation rule, we get

$$\frac{G}{H}(A\tau) = \frac{H}{G}(\tau) \tag{1.42}$$

$$\frac{G}{H}(B\tau) = -\frac{H}{G}(\tau) \tag{1.43}$$

$$\frac{G}{H}(C\tau) = -\frac{G}{H}(\tau). \tag{1.44}$$

As in the $N = 5$ case, any identity of the form

$$G(n\tau)G(m\tau) - H(n\tau)H(m\tau) = \eta - \text{product} \tag{1.45}$$

must satisfy

$$\begin{aligned} G(n\tau)G(m\tau) - H(n\tau)H(m\tau) &\neq 0 \\ \Leftrightarrow \frac{G}{H}(n\tau) &\neq \frac{H}{G}(m\tau) \\ \Leftrightarrow \frac{G}{H}(n\tau) &\neq \frac{G}{H}(Am\tau), \end{aligned}$$

where equation (1.42) is used in the last step. Since $\frac{G}{H}$ is a Hauptmodul for $\Gamma_1(8,2)$, the last inequality holds

$$\Leftrightarrow n\tau \not\sim Am\tau \text{ under } \Gamma_1(8,2)$$

$$\Leftrightarrow n\tau \not\sim m\tau \text{ under the coset } A\Gamma_1(8,2).$$

Similarly, equation (1.43) corresponds to identities of the form

$$G(n\tau)G(m\tau) + H(n\tau)H(m\tau) = \eta - \text{product}, \tag{1.46}$$

and equation (1.44) corresponds to identities of the form

$$G(n\tau)H(m\tau) + H(n\tau)G(m\tau) = \eta - \text{product}. \tag{1.47}$$

Because the only powers of x on the right hand side of (1.28) are multiples of 4, this identity has the following combinatorial interpretation: If $n \not\equiv 0 \pmod 4$, then the number of partitions of n into parts which are (7 times something $\equiv \pm 1$) or something $\equiv \pm 3 \pmod 8$ is equinumerous with the number of partitions of $n-3$ into parts which are (7 times something $\equiv \pm 3$) or something $\equiv \pm 1 \pmod 8$.

N=12: The genus of $X_1(12)$ is 0, and a Hauptmodul is given by $\eta_{12,1}/\eta_{12,5}$. Since $[\Gamma_0(12) : \Gamma_1(12)] = \frac{1}{2}\phi(12) = 2$, there are only two cosets, each again giving rise to all possible identities of a certain form. Using these two cosets, we find the identities given in the previous section. Here we again find an identity (1.37) with a combinatorial interpretation, because there are only even powers of x on the right hand side. The result reads as follows: If n is odd, then the number of partitions of n into parts which are (3 times something $\equiv \pm 1$) or something $\equiv \pm 5 \pmod{12}$ is equinumerous with the number of partitions of $n-2$ into parts which are (3 times something $\equiv \pm 5$) or something $\equiv \pm 1 \pmod{12}$.

N=13: Here the genus of $X_1(13)$ is not 0, but we can go up to the subgroup

$$\Gamma_R(13) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(13) \;\middle|\; \left(\frac{a}{13}\right) = \left(\frac{d}{13}\right) = 1 \right\}, \tag{1.48}$$

which does give a Riemann surface of genus 0. Let $G(n) := \eta_{13,1}^{-1}\eta_{13,3}^{-1}\eta_{13,4}^{-1}(n\tau)$ and $H(n) := \eta_{13,2}^{-1}\eta_{13,5}^{-1}\eta_{13,6}^{-1}(n\tau)$. Notice that the indices in $G(n)$ are the quadratic residues of 13 and the indices in $H(n)$ are the quadratic nonresidues of 13. It turns out that $G(n)/H(n)$ is a Hauptmodul for $\Gamma_R(13)$. Because $[\Gamma_0(13) : \Gamma_R(13)] = 2$, there are two cosets, giving rise to two possible collections of identities. The only identity we find using these two cosets is (1.38), which is particularly simple and has the following curious combinatorial interpretation: The number of partitions of n into quadratic residues of 13 and 3 times quadratic non-residues of 13 is equinumerous with the number of partitions of $n-2$ into quadratic non-residues of 13 and 3 times quadratic residues of 13.