CHAPTER 2

Lacunarity

The question of when the Fourier coefficients of a modular cusp form have zero density has been shown by J.P. Serre and K. Ribet to be equivalent to the representability of the form as a linear combination of Hecke forms over an imaginary quadratic field. In this chapter we prove that a certain class of Dedekind eta products contains only finitely many lacunary forms, and some of their associated Hecke forms are found.

2.1 Some Background

Definition. A power series is called *lacunary* if the arithmetic density of its non-zero coefficients is zero. More precisely, $f(z) = \sum_{k=0}^{\infty} c_k z^k$ is lacunary if

$$\lim_{n\to\infty} \frac{\#\{k \mid k \leq n \text{ } and \text{ } c_k \neq 0\}}{n} = 0.$$

We begin by recalling some results from [GS87]. An η-product is a function $f(\tau)$ of the form

$$f(\tau) = \prod_{\delta \mid N} \eta(\delta\tau)^{r_\delta}, \tag{2.1}$$

where $r_\delta \in Z$. We suppose henceforth that the weight $k = \frac{1}{2}\sum_{\delta|N} r_\delta$ is an integer. Put

$$\Delta = \prod_{\delta|N} \delta^{r_\delta}, \tag{2.2}$$

$$\frac{1}{24}\sum_{\delta|N} \delta r_\delta = \frac{c}{e}, \tag{2.3}$$

and $$\frac{1}{24}\sum_{\delta|N} \frac{N}{\delta} r_\delta = \frac{c_0}{e_0}, \tag{2.4}$$

where the last two fractions are in lowest terms. Let $M = Nee_0$ and $\varepsilon(p) = \left(\frac{(-1)^k \Delta}{p}\right)$ for primes p not dividing M. Then

Theorem 2.1 $F(\tau) := f(e\tau)$ *is a modular form of weight* k *and Nebentypus* ε *on* $\Gamma_0(M)$.

In this paper we will deal with the case $N = 2, r_1 = r$ and $r_2 = s$, so $\Delta = 2^s$ and $\varepsilon(p) = \left(\frac{(-1)^k 2^s}{p}\right)$.

We will extend a paper of Serre [Ser85] which determines all even powers of the Dedekind η-function whose Fourier expansions are lacunary. Recall that $\eta(\tau)$ is a cusp form of weight $\frac{1}{2}$ on $\Gamma_0^0(24, 24)$. It is known that $\eta(\tau)^r$ is lacunary when $r = 1$ or 3. Indeed we have the classical identities of Euler and Jacobi:

$$\prod_{m=1}^{\infty}(1 - x^m) = \sum_{n=-\infty}^{\infty} (-1)^n x^{\frac{3n^2+n}{2}} \tag{2.5}$$

$$\prod_{m=1}^{\infty}(1 - x^m)^3 = \sum_{n=0}^{\infty}(-1)^n(2n+1)x^{\frac{n^2+n}{2}}. \tag{2.6}$$

No other odd values of r are known for which $\eta(\tau)^r$ is lacunary. On the other hand, if r is even and positive, then $\eta(\tau)^r$ has weight $\frac{r}{2} \in Z$, so the theory of modular forms of integral weight comes into play. In fact,

Theorem 2.2 *(Serre) Let $r \geq 0$ be even.*

Then $\eta(\tau)^r$ is lacunary $\Longleftrightarrow r = 2, 4, 6, 8, 10, 14$ or 26.

The implication $\Longleftarrow$ is handled by exhibiting $\eta(\tau)^r$ as a linear combination of Hecke character forms, and for the implication $\Longrightarrow$, Serre shows that for $r \neq 2, 4, 6, 8, 10, 14$ or 26, $\eta(e\tau)^r := f_r(\tau)$ is not annihilated by T_{11}, where e =12,6,4,3,12,12, or 12 respectively. Hence $f_r \notin S_{CM}(N, k, \varepsilon)$, implying that f_r is not lacunary by a theorem of Serre [Ser81]. Similar ideas will be used in this paper. For later use we quote this theorem here:

Theorem 2.3 *Let $f \in M(N, k, \varepsilon)$, with $k \geq 2$, and put $f(x) := \sum c_k x^k$,*

$M_f(T) := \#\{k \mid 0 \leq k \leq T$ and $c_k \neq 0\}$. Then

(i) If $f \notin S_{cm}(N, k, \varepsilon)$, one has $M_f(T) \asymp T$ for $T \to \infty$.

(ii) If $f \in S_{cm}(N, k, \varepsilon)$, and $f \neq 0$, one has $M_f(T) \asymp T/(logT)^{\frac{1}{2}}$ for $T \to \infty$.

2.2 The Problem

We will generalize this to $\eta(\tau)^r\eta(2\tau)^s$, a reasonable next case in view of the fact that

$$\theta(-x) := \sum_{-\infty}^{\infty}(-x)^{n^2} = 1 + 2\sum_{n=1}^{\infty}(-1)^n x^{n^2}$$

is of this type. Namely

$$\theta(-x) = \prod_{n=1}^{\infty}(1-x^n)^2(1-x^{2n})^{-1} = \eta(\tau)^2\eta(2\tau)^{-1}.$$

B. Gordon [GS87] has found the following 34 pairs (r, s) for which $\eta(\tau)^r\eta(2\tau)^s$ is lacunary:

(1,1)	(2,2)	(3,3)	(5,5)	(1,3)	(3,1)	(1,5)	(5,1)
(3,7)	(7,3)	(3,-1)	(-1,3)	(5,-1)	(-1,5)	(6,-2)	(-2,6)
(7,-1)	(-1,7)	(7,-3)	(-3,7)	(8,-2)	(-2,8)	(9,-3)	(-3,9)
(11,-5)	(-5,11)	(2,4)	(4,2)	(-2,4)	(4,-2)	(10,-4)	(-4,10)
(14,-4)	(-4,14).						

One immediately notices that if (r, s) is lacunary then so is (s, r). This fact emerges when one applies the canonical involution $\tau \to \frac{-1}{N\tau}$ to the Riemann surface on which the appropriate form lives [GS87]. We will prove the following

Theorem 2.4 *Let $r + s \in 2Z^+$. Then $\eta(\tau)^r\eta(2\tau)^s$ is lacunary for only a finite number of pairs (r, s).*

The condition $r + s \in 2Z^+$ is assumed because then the weight of $\eta(\tau)^r\eta(2\tau)^s$ is $k = \frac{1}{2}(r + s) \in Z^+$,so that we can use the theory of forms of integraL weight. It is of interest to note that theorem 2.4 holds for $\eta(\tau)^r\eta(3\tau)^s$ as well, and we conjecture that it holds for any $\prod_{\delta|N} \eta(\delta\tau)^{r_\delta}$, where N is fixed.

We now define the Fourier coefficients $a_{r,s}(n)$:

$$\eta(\tau)^r\eta(2\tau)^s = x^{\frac{r+2s}{24}} \prod_{n=1}^{\infty}(1 - x^n)^r(1 - x^{2n})^s$$

$$:= x^{\frac{r+2s}{24}} \sum_{n=0}^{\infty} a_{r,s}(n)x^n.$$

Let $\frac{r+2s}{24} = \frac{c_{r,s}}{e_{r,s}}$ in lowest terms. We write c and e instead of $c_{r,s}$ and $e_{r,s}$ if the subscripts are clear from context. Put $f_{r,s}(\tau) = \eta(\tau)^r\eta(2\tau)^s$ and $F_{r,s}(\tau) = f(e_{r,s}\tau)$

The $b_{r,s}(n)$'s are defined by

$$F_{r,s}(\tau) = \eta(e\tau)^r \eta(e\tau)^s = x^c \sum_{n=0}^{\infty} b_{r,s}(n) x^{ne}$$
$$= \sum_{n=0}^{\infty} b_{r,s}(n) x^{c+24n}.$$

Note: If the b's are lacunary then the a's are lacunary and conversely. It is known [Rib77] that every CM-form is a linear combination of Hecke forms

$$H_C(\tau) := \sum_{(\mathfrak{a},\mathfrak{f})=1} C(\mathfrak{a}) x^{\delta \mathcal{N}(\mathfrak{a})}, \tag{2.7}$$

where $\mathcal{K}$ is a quadratic imaginary field, $\mathcal{N}(\mathfrak{a})$ is the norm of the fractional ideal $\mathfrak{a}$, and $C(\mathfrak{a})$ is a Hecke character (also known as a Grössencharacter) with conductor $\mathfrak{f}$. Furthermore,

$$\delta | d | \mathcal{N}(\mathfrak{f}) \Big| N, \tag{2.8}$$

where d is the discriminant of $\mathcal{K}$. In the present case, $N = 2^\alpha 3^\beta$, and therefore $d = -3, -4, -8$ *or* -24, i.e. $\mathcal{K} = \mathbb{Q}(\sqrt{-3})$, $\mathbb{Q}(\sqrt{-1}$, $\mathbb{Q}(\sqrt{-2})$ or $\mathbb{Q}(\sqrt{-6})$, respectively. The rational primes $p \equiv 23 \pmod{24}$ are inert in all these fields and therefore the Hecke form (2.4) contains no terms in x^p; so if $f(\tau)$ is a CM-form, then $f(\tau) \mid T_p = 0$ for all such primes p. We have therefore established the following

Lemma 2.1 *If $f(\tau) \mid T_p \neq 0$ for some prime $p \equiv 23$ (mod 24), then f is not a CM-form and hence not lacunary.* □

It is now possible to reformulate theorem 2.4 as:

Theorem 2.5 *Theorem 2.4 is equivalent to $a_{r,s}(m)$ vanishing for only finitely many pairs (r,s) such that $0 \le m \le 22$ and $m \equiv -r-2s$ (mod 23).*

Proof: We apply the Hecke operator T_{23}:

$$F_{r,s} \mid T_{23} = \sum_{\substack{n \\ c+24n\equiv 0 (\mathrm{mod}\ 23)}} b_{r,s}(n)x^{\frac{c+24n}{23}} + \varepsilon(23)23^{k-1}\sum_{n=0}^{\infty} b_{r,s}(n)x^{23(c+24n)}.$$

Let m be the smallest integer n in the sum such that $c+24n \equiv 0$ (mod 23). Then we have $m \equiv -c$ (mod 23) and $0 \le m \le 22$. Notice that the second sum has exponents which are $\ge 23c > (c+24m)/23$ when $0 \le m \le 22$ and $c > 1$. Indeed $(23^2-1)c = (24)(22)c > (24)(22)$ when $c > 1$. This implies $23^2c > c+(24)(22) \ge c+24m$. This observation means that the lowest 23 exponents occur in the first sum.

Hence $F_{r,s} \mid T_{23} = b_{r,s}(m)x^{\frac{(c+24m)}{23}} +$ higher order terms. Therefore, by the proposition, to show that $F_{r,s} \mid T_{23} = 0$ for only finitely many pairs (r,s) it suffices to show that $b_{r,s}(m) = 0$ for only finitely many pairs (r,s) such that $0 \le m \le 22$ and $m \equiv -c$ (mod 23). Equivalently, $a_{r,s}(m) = 0$ for only finitely many pairs (r,s) such that $0 \le m \le 22$ and $m \equiv -r-2s$ (mod 23), and the theorem is proved. □

We notice that for a fixed m, $a_{r,s}(m)$ is a polynomial in r and s. We now show that these polynomials, (viewed as varieties over C) for $0 \le m \le 22$, have only finitely many integer points. We use the following theorem of Siegel:

Theorem 2.6 *Let V be an irreducible variety in $P^2(\mathbb{C})$. If the genus of V is greater than or equal to 1, then V has only a finite number of integer points.*

2.3 The Polynomials

It turns out that all 22 polynomials $a_{r,s}(m)$ are irreducible (we used the Maple symbolic manipulation program), up to a factor of r for odd m, which we may neglect since $r = 0$ means we are back in the Serre case, with τ replaced by 2τ. For future reference, the polynomials are:

$$a_{r,s}(0) := 1$$

$$a_{r,s}(1) := -r$$

$$2!a_{r,s}(2) := r^2 - 3r - 2s.$$

$$3!a_{r,s}(3) := 9r^2 - 8r - r^3 + 6rs.$$

$$4!a_{r,s}(4) := 36rs - 12r^2s - 36s + 12s^2 - 18r^3 + 59r^2 - 42r + r^4$$

$$5!a_{r,s}(5) := 340rs - 60rs^2 + 30r^4 - 215r^3 + 450r^2 - 144r - r^5 - 180r^2s + 20r^3s$$

$$6!a_{r,s}(6) := -1440r - 960s + 2880rs - 2475r^3 + 3394r^2 - 2310r^2s + 1080s^2 - 540rs^2 + 565r^4 - 45r^5 - 120s^3 + 540r^3s + 180r^2s^2 - 30r^4s + r^6$$

$$7!a_{r,s}(7) := 22848rs - 5760r - 28294r^3 + 30912r^2 - 30240r^2s - 10920rs^2 + 9345r^4 - 1225r^5 + 10290r^3s + 3780r^2s^2 - 1260r^4s + 42r^5s + 840rs^3 - 420r^3s^2 + 63r^6 - r^7$$

$$8!a_{r,s}(8) := 267120rs - 75600r - 70560s + 293292r^2 - 365624r^2s + 99120s^2 - 126000rs^2 + 183960r^3s + 79800r^2s^2 - 340116r^3 + 147889r^4 - 27720r^5 - 30240s^3 - 34160r^4s + 2520r^5s + 10080rs^3 - 15120r^3s^2 + 2338r^6 - 84r^7 - 3360r^2s^3 + 840r^4s^2 - 56r^6s + 1680s^4 + r^8.$$

$9!a_{r,s}(9) := 2348064rs-524160r+3032208r^2-4992624r^2s-1835568rs^2+3093048r^3s+1496880r^2s^2 - 4335596r^3 + 2341332r^4 - 579369r^5 - 808920r^4s + 92736r^5s + 352800rs^3 - 415800r^3s^2 + 69552r^6 - 4074r^7 - 90720r^2s^3 + 45360r^4s^2 - 4536r^6s + 108r^8 - 1512r^5s^2 + 72r^7s + 10080r^3s^3 - 15120rs^4 - r^9.$

$10!a_{r,s}(10) := 35683200rs - 6531840r - 4354560s + 2520r^6s^2 - 90r^8s + 36290736r^2 - 67124520r^2s + 13608000s^2 - 26535600rs^2 + 52950240r^3s + 26394480r^2s^2 - 57773700r^3 + 38049920r^4 - 11744775r^5 - 6501600s^3 - 17783010r^4s + 2835000r^5s + 5140800rs^3 - 10319400r^3s^2+1857513r^6-154350r^7-2847600r^2s^3+1650600r^4s^2-217980r^6s+907200s^4+6630r^8 - 25200s^3r^4 - 113400r^5s^2 + 7560r^7s + r^{10} + 453600r^3s^3 - 226800rs^4 - 135r^9 - 30240s^5 + 75600r^2s^4.$

$11!a_{r,s}(11) := 330992640rs - 43545600r - 3960r^7s^2 - r^{11} - 277200r^3s^4 + 249480r^6s^2 - 11880r^8s + 110r^9s + 433762560r^2 - 1005143040r^2s - 375186240rs^2 + 920047480r^3s + 494136720r^2s^2 - 831170736r^3 + 636593100r^4 - 237810320r^5 - 381870720r^4s + 78727110r^5s + 119417760rs^3 - 235675440r^3s^2 + 47002725r^6 - 5133513r^7 - 69854400r^2s^3 + 51975000r^4s^2 - 8399160r^6s + 311850r^8 - 1663200s^3r^4 - 5349960r^5s^2 + 460020r^7s + 165r^{10} + 16909200r^3s^3 - 12196800rs^4 - 10230r^9 + 55440r^5s^3 + 332640rs^5 + 2494800r^2s^4.$

$12!a_{r,s}(12) := -1117670400r - 958003200s + 5815877760rs + 5925016800r^2 - 15375560112r^2s + 2257960320s^2 - 6641157600rs^2 + 16511109120r^3s + 8922022560r^2s^2 - 12532005288r^3 + 11101159036r^4 - 4872036510r^5 - 1646568000s^3 + 1163220575r^6 - 8194076220r^4s + 2084197500r^5s +$

$2075673600rs^3 - 5373300240r^3s^2 + 1491349860r^4s^2 - 158764914r^7 -$

$1688591520r^2s^3 - 287741916r^6s - 214552800rs^4 + 543866400r^3s^3 -$

$209563200r^5s^2 + 21871080r^7s + 108939600r^2s^4 + 14885640r^6s^2 -$

$77616000r^4s^3 + 375883200s^4 + 12649263r^8 - 585090r^9 - 29937600s^5 +$

$15125r^10 - 892980r^8s + 4989600r^5s^3 + 5987520rs^5 + 17820r^9s - 498960r^7s^2 -$

$198r^{11} - 14968800r^3s^4 + 665280s^6 - 110880r^6s^3 - 1995840r^2s^5 + 5940r^8s^2 +$

$831600r^4s^4 + r^{12} - 132r^{10}s.$

$13!a_{r,s}(13) := -6706022400r + 65267804160rs + 83648747520r^2 -$

$248712647040r^2s - 94030485120rs^2 + 308640988656r^3s + 175737276000r^2s^2 -$

$201002619168r^3 + 201684764424r^4 - 102004870396r^5 + 28660164390r^6 -$

$177693310080r^4s + 54120194700r^5s + 43071462720rs^3 - 120955005600r^3s^2 +$

$41234553360r^4s^2 - 4722571139r^7 - 40614013440r^2s^3 - 9226477260r^6s -$

$7273506240rs^4 + 16053317280r^3s^3 - 7368821460r^5s^2 + 907340148r^7s +$

$3308104800r^2s^4 + 713512800r^6s^2 - 3091888800r^4s^3 + 469839942r^8 -$

$28445703r^9 + 1033890r^{10} - 51428520r^8s + 291170880r^5s^3 +$

$458377920rs^5 + 1621620r^9s - 36808200r^7s^2 - 21593r^{11} - 724323600r^3s^4 -$

$12972960r^6s^3 - 77837760r^2s^5 + 926640r^8s^2 + 64864800r^4s^4 + 234r^{12} - 25740r^{10}s -$

$2162160r^5s^4 + 205920r^7s^3 - 8580r^9s^2 + 156r^{11}s - 8648640rs^6 + 8648640r^3s^5 - r^{13}.$

$14!a_{r,s}(14) := -149448499200r - 99632332800s + 1406476431360rs +$

$1335385128960r^2 - 4295166273216r^2s + 534693519360s^2 - 1940806707840rs^2 +$

$5946163999776r^3s + 3406539007872r^2s^2 - 3401283910752r^3 + 3821401891944r^4 -$

$2194635751308r^5 - 489409240320s^3 + 711131447026r^6 - 3942001815752r^4s +$

$1399393915920r^5s + 880863984000rs^3 - 2778946410240r^3s^2 + 1118493175800r^4s^2 -$

$137914387611r^7 - 947717971200r^2s^3 - 285595548238r^6s - 148021473600rs^4$

$+ 464466562560r^3s^3 - 244375431300r^5s^2 + 34699532868r^7s + 104906561760r^2s^4$

$+ 30192798636r^6s^2 - 111109438440r^4s^3 + 161643081600s^4 + 16556785817r^8$

$- 1250368119r^9 - 21189168000s^5 + 59372313r^{10} - 2543967426r^8s +$

$14075661600r^5s^3 + 9444314880rs^5 + 111351240r^9s - 2126484360r^7s^2 -$

$1738737r^{11} - 28832403600r^3s^4 + 1089728640s^6 - 933332400r^6s^3 -$

$4510265760r^2s^5 + 82882800r^8s^2 + 3758554800r^4s^4 + 29939r^{12} -$

$2788786r^{10}s - 227026800r^5s^4 + 30270240r^7s^3 - 1621620r^9s^2 + 36036r^{11}s$

$- 181621440rs^6 + 544864320r^3s^5 - 273r^{13} - 182r^{12}s + 60540480r^2s^6 -$

$30270240r^4s^5 + 5045040r^6s^4 + r^{14} - 17297280s^7 - 360360r^8s^3 + 12012r^{10}s^2.$

$15!a_{r,s}(15) := -2092278988800r + 15546595921920rs + 90810720r^5s^5 +$

$20323375994880r^2 - 77293744819200r^2s - 29344426675200rs^2 +$

$119980335558720r^3s + 70847669692800r^2s^2 - 60929911689984r^3 +$

$75623303901600r^4 - 48648442726520r^5 + 17905744436580r^6 -$

$89521050659040r^4s + 36460066482840r^5s + 17581820256000rs^3 -$

$64910482116480r^3s^2 + 30269764324800r^4s^2 - 4006881036158r^7 -$

$123365079337600r^2s^3 - 8691137254800r^6s - 4247433590400rs^4 +$

$13056726883200r^3s^3 - 7849269628200r^5s^2 + 1264029576810r^7s +$

$3099369873600r^2s^4 + 1186044759900r^6s^2 - 3810417811200r^4s^3 +$

$566030289825r^8 - 51550612685r^9 + 3049771725r^{10} -$

$114509615220r^8 s + 605871466200r^5 s^3 + 439887127680rs^5 +$

$6481104630r^9 s - 106746379740r^7 s^2 - 116473357r^{11} - 1062636775200r^3 s^4 -$

$54032378400r^6 s^3 - 163459296000r^2 s^5 + 5691886200r^8 s^2$

$+ 182756574000r^4 s^4 + 2805075r^{12} - 225225000r^{10} s - 15967551600r^5 s^4$

$+ 2641438800r^7 s^3 - 172972800r^9 s^2 + 4583670r^{11} s - 18767548800rs^6$

$+ 33145912800r^3 s^5 - 40495r^{13} - 49140r^{12} s + 2724321600r^2 s^6$

$- 2724321600r^4 s^5 + 681080400r^6 s^4 + 315r^{14} - 64864800r^8 s^3 + 2702700r^{10} s^2 +$

$259459200rs^7 - 16380r^{11} s^2 + 600600r^9 s^3 - r^{15} - 10810800r^7 s^4 + 210r^{13} s - 302702400r^3 s^6.$

$16! a_{r,s}(16) := -40537905408000r - 39230231040000s + 387277104614400rs$

$-1143429812726400r^3 + 376785057196800r^2 - 1487211185606400r^2 s +$

$152194615372800s^2 - 699351516864000rs^2 + 1560949651635984r^4$

$-1113697174699200r^5 - 176492450534400s^3 + 2508103324621440r^3 s +$

$1493889875936640r^2 s^2 - 2092744928426880r^4 s + 962960425146720r^5 s +$

$412723825113600rs^3 - 1550918820971520r^3 s^2 + 825866740739040r^4 s^2$

$-561265462517760r^2 s^3 - 263323750850160r^6 s - 99020009088000rs^4 +$

$370689532012800r^3 s^3 - 248247081482400r^5 s^2 + 44764604604720r^7 s +$

$459649616109320r^6 - 116869372980360r^7 + 76742323257600s^4 + 19052174174033r^8$

$-2041046092800r^9 - 14384418048000s^5 + 90757876809600r^2 s^4 +$

$44561198201520r^6 s^2 - 126534566558400r^4 s^3 - 4854285733440r^8 s +$

$24417489096000r^5 s^3 + 10265243788800rs^5 + 339461282160r^9 s +$

$145516059260r^{10} - 6915788880r^{11} - 4909499955360r^7s^2$

$-37648308297600r^3s^4 + 1213231219200s^6 - 2746983919680r^6s^3$

$-6579297204480r^2s^5 + 334876061520r^8s^2 + 7986348770400r^4s^4$

$-15248032800r^{10}s + 216834982r^{12} - 926269344000r^5s^4 +$

$181016035200r^7s^3 - 13945932000r^9s^2 + 430269840r^{11}s$

$-443156313600rs^6 + 1580106528000r^3s^5 - 4368000r^{13} - 7250880r^{12}s +$

$202205203200r^2s^6 + 53620r^{14} - 43589145600s^7 - 191307916800r^4s^5 +$

$57816158400r^6s^4 - 6760353600r^8s^3 + 338978640r^{10}s^2 - 4324320r^{11}s^2 +$

$6227020800rs^7 - 360r^{15} + 129729600r^9s^3 + 10897286400r^5s^5 - 1816214400r^7s^4 +$

$65520r^{13}s - 21794572800r^3s^6 - 240r^{14}s - 2075673600r^2s^7 +$

$1210809600r^4s^6 + r^{16} + 518918400s^8 - 242161920r^6s^5 + 21621600r^8s^4$

$-960960r^{10}s^3 + 21840r^{12}s^2$

$17!a_{r,s}(17) := 5110497739468800rs - 376610217984000r$

$-22572470529457920r^3 + 6493118120294400r^2 - 28944951261649920r^2s$

$-11068998799703040rs^2 + 33612562257132672r^4 - 26348214301856784r^5 +$

$54845297684736768r^3s + 33233795027596800r^2s^2 - 50398593297736320r^4s +$

$25901038432097920r^5s + 8308279408665600rs^3 - 38152801231359360r^3s^2 +$

$22802868521072640r^4s^2 - 14362097291942400r^2s^3 - 8009041670229600r^6s$

$-2641928947257600rs^4 + 10522640876267520r^3s^3 - 7826069103172320r^5s^2 +$

$1562565474726256r^7s + 12069423054212160r^6 - 3443059993897160r^7 +$

$637855242288264r^8 - 78849375466433r^9 + 2705165208345600r^2s^4 +$

$1631042325712800r^6s^2-4153890649708800r^4s^3-198106456307760r^8s+$

$944172725496000r^5s^3+389868237158400rs^5+16582252017760r^9s+$

$6609649984080r^{10}-378267538220r^{11}-213572965285680r^7s^2$

$-1286266456675200r^3s^4-128900671099200r^6s^3-231221528778240r^2s^5+$

$17847078369120r^8s^2+330276772425600r^4s^4-920081962800r^{10}s+$

$14741746656r^{12}-47600810056800r^5s^4+10745263488960r^7s^3$

$-951553883280r^9s^2+33542722304r^{11}s-27145866908160rs^6+$

$70878130202880r^3s^5-385985782r^{13}-783029520r^{12}s+$

$8521677964800r^2s^6+6597360r^{14}-11053480838400r^4s^5+$

$3952082534400r^6s^4-542529187200r^8s^3+31733301600r^{10}s^2$

$-630062160r^{11}s^2+835112678400rs^7-69700r^{15}+15927912000r^9s^3+$

$905685580800r^5s^5-184028644800r^7s^4+11100320r^{13}s$

$-1626117292800r^3s^6-85680r^{14}s-105859353600r^2s^7+$

$123502579200r^4s^6+408r^{16}-37050773760r^6s^5+4410806400r^8s^4$

$-245044800r^{10}s^3+6683040r^{12}s^2+11762150400r^3s^7+$

$1485120r^{11}s^3-8821612800rs^8-4116752640r^5s^6$

$-28560r^{13}s^2-40840800r^9s^4-r^{17}+272r^{15}s+588107520r^7s^5$

$18!a_{r,s}(18):=-13871809695744000r-9247873130496000s+$

$13978007799275520 0rs-4680134634414758 40r^3+132672192555571200r^2$

$-620280438160919040r^2s+534979298101248 00s^2-300179546739072000rs^2+$

$753265741573302912r^4-644649799518037296r^5-764938983384576 00s^3+$

$12406054585744051 20r^3s + 761958469445575680r^2s^2 - 1252469272848806304r^4s +$

$711724121298852480r^5s + 220566220255180800rs^3 - 962452310446972800r^3s^2 +$

$641304953465904000r^4s^2 - 363766470731838720r^2s^3 - 245919762452997360r^6s$

$-69528555768672000rs^4 + 303737574886986240r^3s^3 - 247599973947886560r^5s^2 +$

$54276667971926400r^7s + 32483351516216 3344r^6 - 102896818979343480r^7 +$

$41308648680499200s^4 + 21393928537769424r^8 - 3005469776175867r^9$

$-10221937972646400s^5 + 78918340419959040r^2s^4 + 58924973286859680r^6s^2$

$-135920942078201280r^4s^3 - 7899950003792418r^8s + 35656284084220800r^5s^3 +$

$10226013557760000rs^5 + 773144623920240r^9s + 290643272718513r^{10}$

$-19547135742420r^{11} - 8966792838603600r^7s^2 - 43922408747817600r^3s^4 +$

$1227121626931200s^6 - 5744943183339360r^6s^3 - 8300659513386240r^2s^5 +$

$890923478187600r^8s^2 + 13105107154339200r^4s^4 - 51341004978120r^{10}s +$

$916590073516r^{12} - 2277812100564000r^5s^4 + 581817563046720r^7s^3$

$-58229591500080r^9s^2 + 2311299310320r^{11}s - 714709425830400rs^6 +$

$2992665098142720r^3s^5 - 29796417378r^{13} - 69667406172r^{12}s +$

$424280760583680r^2s^6 + 660951942r^{14} - 71878501094400s^7$

$-574537086002880r^4s^5 + 236342254468320r^6s^4 - 37236101142240r^8s^3 +$

$2487731566320r^{10}s^2 - 67816148400r^{11}s^2 + 22230464256000rs^7$

$-9703260r^{15} + 1482030950400r^9s^3 + 616895383104 00r^5s^5$

$-14681369102400r^7s^4 + 1366681680r^{13}s - 90589141843200r^3s^6$

$-16517880r^{14}s - 9791990208000r^2s^7 + 10312465363200r^4s^6 +$

$89148r^{16} + 1905468364800s^8 - 3649501215360r^6s^5 +$

$527091364800r^8s^4-35016901920r^{10}s^3+1119409200r^{12}s^2+$

$952734182400r^3s^7+441080640r^{11}s^3-238183545600rs^8$

$-555761606400r^5s^6-10024560r^{13}s^2-9924314400r^9s^4$

$-459r^{17}+110160r^{15}s+111152321280r^7s^5$

$-306r^{16}s+r^{18}-17643225600s^9+79394515200r^2s^8$

$-52929676800r^4s^7+12350257920r^6s^6-1323241920r^8s^5+$

$73513440r^{10}s^4-2227680r^{12}s^3+36720r^{14}s^2$

$19!a_{r,s}(19):=1963197736567603200rs-128047474114560000r-$

$1012412497960617984 0r^3+25133514500247552 00r^2-1312394450860265472 0r^2s$

$-502588081290903552 0rs^2+1758130934353299532 8r^4-1630716649229796249 6r^5$

$+2928681030103374182 4r^3s+1809985617416729088 0r^2s^2-3212358269246015846 4r^4s$

$+2001780055184300668 8r^5s+4555102063832893440rs^3-2510427090355889356 8r^3s^2$

$+1838941052129423424 0r^4s^2-9749670852705638400r^2s^3-7654430048712370560r^6s$

$-1808982298793779200rs^4+8900165690300102400r^3s^3-7907244660671690880r^5s^2+$

$1888702623232156368r^7s+8973226283666266512r^6-3128796899659725904r^7$

$+722588147763843816r^8-113940510141447912r^9+2409750117642067200r^2s^4$

$+2119378099720321440r^6s^2-4463920200326146560r^4s^3-311072446083567168r^8s$

$+1330948288091822400r^5s^3+351357833463436800rs^5+34955529492283686r^9s$

$+12520831224647889r^{10}-971152877510673r^{11}-368445269134758816r^7s^2$

$-1483248899313941760r^3s^4-247742526138307200r^6s^3-291422077650063360r^2s^5$

$+42566761405087920r^8s^2+510431144746723200r^4s^4-2713862641192560r^{10}s$

$$+ 53486430829884r^{12} - 103831916146358400r^5s^4 + 29638922930416800r^7s^3$$

$$-3309151062531600r^9s^2 + 146187484304472r^{11}s - 34738043290030080rs^6$$

$$+ 121704758623768320r^3s^5 - 2090372264188r^{13} - 5439181234032r^{12}s +$$

$$17318939584826880r^2s^6 + 57506558886r^{14} - 28029058552535040r^4s^5$$

$$+ 12990348631260000r^6s^4 - 2307353255887680r^8s^3 + 173382734725200r^{10}s^2$$

$$-6057630646704r^{11}s^2 + 1732557696629760rs^7 - 1094076582r^{15}$$

$$+ 116596947186720r^9s^3 + 3712768499119680r^5s^5 - 1012204864550880r^7s^4$$

$$+ 137683694484r^{13}s - 4848439553717760r^3s^6 - 2300250960r^{14}s$$

$$-470650686105600r^2s^7 + 693405230918400r^4s^6 + 13941972r^{16}$$

$$-289329492291840r^6s^5 + 48648989188800r^8s^4 - 3743304364800r^{10}s^3$$

$$+ 137326447440r^{12}s^2 + 85481428032000r^3s^7 + 72588952800r^{11}s^3$$

$$-40226554368000rs^8 - 53735972209920r^5s^6 - 1912806000r^{13}s^2$$

$$-1382787806400r^9s^4 - 112404r^{17} + 23976936r^{15}s +$$

$$12889258462080r^7s^5 - 139536r^{16}s + 513r^{18} + 4525487366400r^2s^8$$

$$-6033983155200r^4s^7 + 2111894104320r^6s^6 - 301699157760r^8s^5$$

$$+ 20951330400r^{10}s^4 - 761866560r^{12}s^3 + 14651280r^{14}s^2$$

$$+ 335221286400rs^9 + 342r^{17}s - r^{19} - 502831929600r^3s^8 + 201132771840r^5s^7$$

$$-33522128640r^7s^6 + 2793510720r^9s^5 - 126977760r^{11}s^4 + 3255840r^{13}s^3 - 46512r^{15}s^2$$

$$20!a_{r,s}(20) := -5109094217170944000r - 4379223614717952000s$$

$$+ 6103079503106826240 0rs - 22977721262513228544 0r^3 + 5657742698042050560 0r^2$$

$$-30915704011318442496 0r^2s + 24330854412645580800s^2 - 15180119836340659200 0rs^2 +$$

$$4263080578912007976 96r^4 - 4264561048053850056 00r^5 - 387339480681753600 00s^3 +$$

$$715737494778430464000r^3s + 450460391857170739200r^2s^2 - 850198174406993240640r^4s +$$

$$577152378844881209280r^5s + 135311669989816320000rs^3 - 671895792626577235200r^3s^2 +$$

$$539506520030077680960r^4s^2 - 262575955613686694400r^2s^3 - 242189396259858970400r^6s$$

$$-53095235271414336000rs^4 + 265844617219663372800r^3s^3 - 255797920390745678400r^5s^2$$

$$+ 66162375829911816000r^7s + 25462401322289579064 0r^6 - 97003598589686195760r^7$$

$$+ 2551028625963417600 0s^4 + 24669235359489920536r^8 - 4321657983197020050r^9$$

$$-7874197167957120000s^5 + 72274200919381843200r^2s^4 + 76372520945661158400r^6s^2$$

$$-148015783787578761600r^4s^3 - 12186731417793122580r^8s + 494780941753914816 00r^5s^3$$

$$+ 10257849805621017600rs^5 + 1549912622942429460r^9s + 533059026033824355r^{10}$$

$$-46974405357323730r^{11} - 14962307005007762400r^7s^2 - 50639405635743724800r^3s^4$$

$$+ 1245355794729446400s^6 - 10473884527805899200r^6s^3 - 10150323086042979840r^2s^5$$

$$+ 1975667569604405820r^8s^2 + 19641153610055577600r^4s^4 - 138169816941456300r^{10}s$$

$$+ 2982559682950341r^{12} - 4594096478547576000r^5s^4 + 1449085623648062400r^7s^3$$

$$-178652760540883200r^9s^2 + 8700621098598000r^{11}s - 1013497980663168000rs^6$$

$$+ 4892604958200960000r^3s^5 - 136830847448100r^{13} - 387247492154120r^{12}s$$

$$+ 747896121465292800r^2s^6 + 4521894989470r^{14} - 103482811111680000s^7$$

$$-1304033124246652800r^4s^5 + 674480455198149600r^6s^4 - 133365612516364800r^8s^3$$

$$+ 11137043329354800r^{10}s^2 - 477341144280000r^{11}s^2 + 50806138166784000rs^7$$

$$-106580794500r^{15} + 8206678020340800r^9s^3 + 207115214722416000r^5s^5$$

$$-63447458584910400r^7s^4 + 12094663384440r^{13}s - 238848183551577600r^3s^6$$

$$-260551347960r^{14}s - 28392572515507200r^2s^7 + 42136393841942400r^4s^6$$

$$+ 1757721426r^{16} + 4445034257664000s^{8} - 20016414993294720r^{6}s^{5}$$

$$+ 3835567040104800r^{8}s^{4} - 334969870435200r^{10}s^{3} + 13872542491080r^{12}s^{2}$$

$$+ 5460754755456000r^{3}s^{7} + 8841461428800r^{11}s^{3} - 1191711673152000rs^{8}$$

$$-4223788208640000r^{5}s^{6} - 265383518400r^{13}s^{2} - 146449799496000r^{9}s^{4}$$

$$-19622250r^{17} + 3750030000r^{15}s + 1179140874912000r^{7}s^{5}$$

$$-34050660r^{16}s + 139935r^{18} - 90509747328000s^{9}$$

$$+ 510374408544000r^{2}s^{8} - 589989464064000r^{4}s^{7} + 238174723987200r^{6}s^{6}$$

$$-40813191619200r^{8}s^{5} + 3368508343200r^{10}s^{4} - 143061609600r^{12}s^{3}$$

$$+ 3159327600r^{14}s^{2} + 10056638592000rs^{9} + 174420r^{17}s - 570r^{19}$$

$$-45254873664000r^{3}s^{8} + 30169915776000r^{5}s^{7} - 7039647014400r^{7}s^{6}$$

$$+ 754247894400r^{9}s^{5} - 41902660800r^{11}s^{4} + 1269777600r^{13}s^{3}$$

$$-20930400r^{15}s^{2} - 380r^{18}s - 3352212864000r^{2}s^{9} + 2514159648000r^{4}s^{8}$$

$$+ r^{20} + 670442572800s^{10} - 670442572800r^{6}s^{7} + 83805321600r^{8}s^{6}$$

$$-5587021440r^{10}s^{5} + 211629600r^{12}s^{4} - 4651200r^{14}s^{3} + 58140r^{16}s^{2}$$

$$21!a_{r,s}(21) := 92673061780303872000rs - 77852864261652480000r$$

$$+ 10666131840r^{11}s^{5} - 5391444477776477184000r^{3}$$

$$+ 11882832802265456640000r^{2} - 7062485654138432716800r^{2}s$$

$$-2726068779750680985600rs^{2} + 1074241744872720092928 0r^{4}$$

$$-1152477610402843641369 6r^{5} + 18184875589306488222720r^{3}s +$$

$$11394909148514524416000r^{2}s^{2} - 23230888814613846804480r^{4}s +$$

$$17068263482640002391360r^{5}s + 2880699347952708710400rs^{3}$$

$-1860921062247282029568 0r^3s^2 + 1619725637700487062144 0r^4s^2$

$-740024029049116815360 0r^2s^3 - 780602198277166773984 0r^6s$

$-138251621375196456960 0rs^4 + 812207930603799566592 0r^3s^3$

$-840601714408252721568 0r^5s^2 + 234151994555767691088 0r^7s +$

$742492387740412325520 0r^6 - 307098355073632448984 0r^7 +$

$85365353787515170488 0r^8 - 16471491000734091853 6r^9 +$

$228061734651244166400 0r^2s^4 + 276994691000454903840 0r^6s^2$

$-496605772068715895040 0r^4s^3 - 477661247777904240960 r^8s +$

$184352978680232806080 0r^5s^3 + 335538356063234688000 rs^5 +$

$6795377416455262362 0r^9s + 2257360458933954465 0r^{10}$

$-223267102084067905 5r^{11} - 604763879003531170560 r^7s^2$

$-173618870992416420480 0r^3s^4 - 437992104428041363200 r^6s^3$

$-361547442942227942400 r^2s^5 + 900281573698455748 80r^8s^2 +$

$753822552055353580800 r^4s^4 - 6856835683226108580 r^{10}s +$

$161020940870274030 r^{12} - 1995643806113333376 00r^5s^4 +$

$68952938006626914240 r^7s^3 - 9307605012855409260 r^9s^2 +$

$4957648038840861 00r^{11}s - 43596491722297113600 rs^6 +$

$193257138785184714240 r^3s^5 - 8512825633380341 r^{13}$

$-25790043520790640 r^{12}s + 30308172432035097600 r^2s^6 +$

$330043094684100 r^{14} - 591988556205446400 00r^4s^5 +$

$337064502738403440 00r^6s^4 - 73443953868837552 00r^8s^3 +$

$675082747749009600 r^{10}s^2 - 34433438690654640 r^{11}s^2 +$

$3077238888076953600rs^7 - 9337482544870r^{15} +$

$534666395985321600r^9s^3 +$

$10945664970172732800r^5s^5 - 3726020096873474400r^7s^4 +$

$963244729655880r^{13}s - 11373044091420211200r^3s^6$

$-25590631351320r^{14}s - 13217078062476288800r^2s^7 +$

$23826178095527808000r^4s^6 + 190585071180r^{16}$

$-12685883897887200000r^6s^5 + 2723401841784624000r^8s^4$

$-265995995625360000r^{10}s^3 + 1227863765157120r^{12}s^2 +$

$343210950540057600r^3s^7 + 893603785052160r^{11}s^3$

$-115140466567526400rs^8 - 293205404610518400r^5s^6$

$-30114883126440r^{13}s^2 - 13108458264501600r^9s^4$

$-2749977426r^{17} + 474581819880r^{15}s +$

$930680053362259200r^7s^5 - 5943535920r^{16}s +$

$27112050r^{18} + 275602180613760000r^2s^8$

$-454057232428800000r^4s^7 + 215835577461504000r^6s^6$

$-4276585561248000r^8s^5 + 406246296456000r^{10}s^4$

$-19705678574400r^{12}s^3 + 492283008000r^{14}s^2 +$

$2088428614272000rs^9 + 47425140r^{17}s - 172235r^{19}$

$-4804559087328000r^3s^8 + 3350871978854400r^5s^7$

$-923367033388800r^7s^6 + 117914087491200r^9s^5$

$-7701836032800r^{11}s^4 + 269844019200r^{13}s^3 - 5064459120r^{15}s^2$

$-215460r^{18}s - 211189410432000r^2s^9 + 316784115648000r^4s^8 +$

$630r^{20}-126713646259200r^6s^7+21118941043200r^8s^6$

$-1759911753600r^{10}s^5+79995988800r^{12}s^4$

$-2051179200r^{14}s^3+29302560r^{16}s^2$

$-14079294028800rs^{10}+420r^{19}s-r^{21}+23465490048000r^3s^9$

$-10559470521600r^5s^8+2011327718400r^7s^7$

$-195545750400r^9s^6-341863200r^{13}s^4+$

$6511680r^{15}s^3-71820r^{17}s^2$

$22!a_{r,s}(22) := -1839273918181539840000r-1226182612121026560000s+$

$319822810256212623360000rs+3871805857920r^{11}s^5+$

$537213600r^{14}s^4-133215980629223355494400r^3+$

$29682641812682686464000r^2-181642671299872432742400r^2s+$

$12214141241850003456000s^2-91168970221341031219200rs^2+$

$280493478588578526881280r^4-321696826292601973559808r^5$

$-23404594360556202393600s^3+478425579198190972508160r^3s+$

$306856907200354424094720r^2s^2-654613866542039575514112r^4s+$

$518084325036533389559040r^5s+94599758187886887936000rs^3$

$-528961523413407975475200r^3s^2+498363048472719032507520r^4s^2$

$-212392349701403158425600r^2s^3-256650781773468584003040r^6s$

$-44846551127575701504000rs^4+253221405576801848524800r^3s^3$

$-281162524045843380790080r^5s^2+83940798650133036648960r^7s+$

$222544038268048807560096r^6-99375421479693578585520r^7+$

$$1792556286321056256000 0s^4 + 3000216250416053386256 0r^8$$

$$-6329346454409120642808r^9 - 6696402836726034432000s^5 +$$

$$7132905470741083165440 0r^2s^4 + 1014522182010523652188 80r^6s^2$$

$$-1692580294439878744454 40r^4s^3 - 1880907004524538728091 2r^8s +$$

$$6911245457232007731840 0r^5s^3 + 1080335120800923187200 0rs^5 +$$

$$2964206610025967386080r^9s + 9555224599873758467 86r^{10}$$

$$-1050045166396435359 15r^{11} - 2445792486655640522400 0r^7s^2$$

$$-6041264767603932746880 0r^3s^4 + 1323530370859656960000s^6$$

$$-1823932475169248283840 0r^6s^3 - 1267889619642811925760 0r^2s^5 +$$

$$4059158790479204051640r^8s^2 + 2899532938615720767360 0r^4s^4$$

$$-3346171887473745630 30r^{10}s + 8498543267488366365r^{12}$$

$$-8582307936382897516800r^5s^4 + 3224981962691871782400r^7s^3$$

$$-4732686130738266300 60r^9s^2 + 2738487405986666334 0r^{11}s$$

$$-1399546774246074854400rs^6 + 7670263668127519979520r^3s^5$$

$$-510205794697752153r^{13} - 1635527292307810902r^{12}s +$$

$$1242332500053042478080r^2s^6 + 227852375537196 41r^{14}$$

$$-144552477855334348800s^7 - 2636901548186374909440r^4s^5 +$$

$$1642485311745910632000r^6s^4 - 3910825592029679122 80r^8s^3 +$$

$$3925504204095419946 0r^{10}s^2 - 2329988580895575600r^{11}s^2 +$$

$$988983113900221440 00rs^7 - 755818592808510r^{15} +$$

$$3296909015740660320 0r^9s^3 + 5587562551965698304 00r^5s^5$$

$$-2090855712971169144 00r^7s^4 + 71381472965122320r^{13}s$$

$$-52899105073875540480 0r^3s^6-2267033851610220r^{14}s$$

$$-6758681829557898240 0r^2s^7+12802466357927326080 0r^4s^6+$$

$$18503077089970r^{16}+878125568576256000 0s^8$$

$$-756206908462173168 00r^6s^5+17978258125572544800r^8s^4$$

$$-1941661599994116000r^{10}s^3+98889016045179360r^{12}s^2+$$

$$19394959647961497600r^3s^7+79566685881883200r^{11}s^3$$

$$-3721579790632704000rs^8-18785518046895600000r^5s^6$$

$$-2976164767616040r^{13}s^2-10477639024836984 00r^9s^4$$

$$-330104774538r^{17}+51821369800200r^{15}s+$$

$$6661802056496146560r^7s^5-835615161612r^{16}s+$$

$$4201519476r^{18}-2880623558292480 00s^9+19805835685603968 00r^2s^8$$

$$-31813467192771264 00r^4s^7+17043315111997574 40r^6s^6-38247698697560352 0r^8s^5+$$

$$41011565799204000r^{10}s^4-2235154738536720r^{12}s^3+62370866055960r^{14}s^2+$$

$$68143783099392000rs^9+9186352560r^{17}s-36845655r^{19}$$

$$-346720214576736000r^3s^8+299677773403008000r^5s^7-95672322749203200r^7s^6$$

$$+14054655264249600r^9s^5-1050227338960800r^{11}s^4+41749017710400r^{13}s^3$$

$$-880761697200r^{15}s^2-64913310r^{18}s-28522303153344000r^2s^9$$

$$+35814204185760000r^4s^8+209825r^{20}+4646167029504000s^{10}$$

$$-16184148486105600r^6s^7+3200692842547200r^8s^6-315121976769600r^{10}s^5$$

$$+16670275221600r^{12}s^4-489759732000r^{14}s^3+7905993480r^{16}s^2$$

$$-464616702950400rs^{10}+263340r^{19}s-693r^{21}$$

$$+2323083514752000r^3s^9-1742312636064000r^5s^8$$

$$+464616702950400r^7s^7-58077087868800r^9s^6$$

$$-146659312800r^{13}s^4+3223281600r^{15}s^3-40291020r^{17}s^2$$

$$-462r^{20}s+r^{22}+154872234316800r^2s^{10}-129060195264000r^4s^9$$

$$+38718058579200r^6s^8-28158588057600s^{11}-5531151225600r^8s^7$$

$$+430200650880r^{10}s^6-19554575040r^{12}s^5-8953560r^{16}s^3+87780r^{18}s^2$$

2.4 The Proof

Lemma 2.2 *The polynomials $a_{r,s}(m)$ ($m = 2$ or 3) are never zero for the pairs (r,s) with $r+2s \equiv -m$ (mod 23).*

Proof: Suppose on the contrary that $a_{r,s}(2) = 0$ and that $r+2s \equiv -m = -2$ (mod 23). This implies that $r^2 - 3r - 2s = 0$ and that $r + 2s + 2 \equiv 0$ (mod 23). Adding, we get $r^2 - 2r + 2 \equiv 0$ (mod 23). But this congruence has no solution since its discriminant is -4, a quadratic non-residue of 23.

Similarly, setting $a_{r,s}(3) = 0$ and using $r + 2s \equiv -m = -3$ (mod 23), we get $r^2 - 6r - 6 \equiv 0$ (mod 23). Again, this congruence has no solution since its discriminant is 60, a quadratic non-residue of 23. □

To apply Siegel's theorem , we first homogenize these polynomials with a third variable t. We then look for all singular points on the homogenized curves $a_{r,s,t}(m) = 0$ by setting the three partials equal to zero and solving the simultaneous system. Because of Euler's formula for any homogeneous polynomial P:

$$3P(r,s,t) = r\frac{\partial P}{\partial r} + s\frac{\partial P}{\partial s} + t\frac{\partial P}{\partial t};$$

we need only check that the first two partials are zero.

Lemma 2.3 *There are no finite singular points on the curves* $a_{r,s,t}(4), \ldots, a_{r,s,t}(23)$

Proof: To work in the affine plane, we set $t = 1$. Let $R_1(s) := R(a_{r,s}(m), \frac{\partial a_{r,s}(m)}{\partial s}; r)$ denote the resutant with respect to r. That is, we think of r as the variable and of $a_{r,s}(m)$ and $\frac{\partial a_{r,s}(m)}{\partial s}$ as polynomials over the ring $\mathbf{Z}[s]$. The variable r is now eliminated by the resultant and we get a polynomial in s, namely $R_1(s)$. Similarly, define $R_2(s) := R(\frac{\partial a_{r,s}(m)}{\partial s}, \frac{\partial a_{r,s}(m)}{\partial r}; s)$. Finally, let $R_3 := R(R_1(s), R_2(s)) \in \mathbf{Z}$ be the usual resultant. If the curve $a_{r,s}(m)$ has a finite singular point (r_0, s_0), then $R_1(s_0) = R_2(s_0) = 0$, and hence $R_3 = 0$. Therefore $R_3 \neq 0$ means that the curve has no finite singular points. Moreover, we can carry out this whole procedure mod p, since the resultant is simply a determinant in the coefficients of the two polynomials. If we then get $R_3 \not\equiv 0 \bmod p$, then we know that the curve has no finite singularities. It is found, using Maple, that this is indeed the case. □

Proof of Theorem 2.4: We observed that we must only prove Theorem 2.5. When $t \neq 0$, we are on the affine part of the curve and by lemma 2.3 there are no finite singular points. On the other hand $t = 0$ implies that $r = 0$, because on the curve $a_{r,s,t}(m)$, the only surviving term after setting $t = 0$ is r^m, up to sign. Thus $(0,1,0)$ is the only singular point at infinity on the curves $a_{r,s,t}(m) = 0$.

Unfortunately, $(0,1,0)$ is a multiple singularity for the curve $a_{r,s,t}(m) = 0$. To determine its multiplicity, we use the technique of 'blowing up' the curve at the singularity. The points on the new blown-up curve lying above the singularity are the 'infinitely near points'. We again blow up this new curve at each of these points to get more infinitely near points on yet a new curve. In a finite number of steps the process terminates, and all of these infinitely near points become de-singularized, giving us a non-singular model of the original curve. The genus of the curve is given by [Har77]:

$$g(V) = \binom{n-1}{2} - \sum_P \binom{r_p}{2}, \tag{2.9}$$

where the sum is taken over all points P which are singular and infinitely near to the original singular points, n is the degree of the curve, and r_p is the multiplicity at each of the original singular points or their infinitely near counterparts.

To illustrate by an example, we now calculate the genus of $a_{r,s}(4)$. First we homogenize to get:

$$a_{r,s,t}(4) = 36rst^2 - 12r^2st - 36st^3 + 12s^2t^2 - 18r^3t + 59r^2t^2 - 42rt^3 + r^4$$

Because the only singularity is $(0,1,0)$, setting $s = 1$ will move us into the local coordinates (r,t) on the variety :

$$36rt^2 - 12r^2t - 36t^3 + 12t^2 - 18r^3t + 59r^2t^2 - 42rt^3 + r^4$$

Now we proceed to blow up the curve at the origin by letting $t := r\alpha$. We get

$$36r\alpha^2 - 12r\alpha - 36r\alpha^3 + 12\alpha^2 - 18r^2\alpha + 59r^2\alpha^2 - 42r^2\alpha^3 + r^2$$

after factoring out r^2. This curve still has a double point at the origin, so we blow up again. Let $r := \alpha\beta$. We now get

$$36\alpha\beta - 12\beta - 36\alpha^2\beta + 12 - 18\alpha\beta^2 + 59\alpha^2\beta^2 - 42\alpha^3\beta^2 + \beta^2$$

after factoring out α^2. This curve is finally non-singular, so the singularity at the origin of the original curve has been resolved. The (r, t) curve had a singularity of multiplicity 2 at the origin, so that $r_p = 2$ for this curve. Similarly, the (r, α) curve had a singularity of multiplicity 2 at the origin, so that $r_p = 2$ in this case. Therefore

$$g\left(c_{r,s}(4)\right) = \binom{4-1}{2} - \binom{2}{2} - \binom{2}{2} = 1.$$

It is worth mentioning that, alternatively, one can calculate the genus by finding the Puiseux expansions of the algebraic functions and using perturbation to discern the multiplicity of the singularity. Proceeding to blow up all of the 22 curves, it is found that

$$g\left(c_{r,s}(m)\right) \geq 1 \quad for \quad m \geq 4,$$

implying by theorem 2.6 that they only vanish for finitely many integer points. Furthermore, $c_{r,s}(2)$ and $c_{r,s}(3)$ never vanish by lemma 2.2. Thus, the theorem is proved. □

2.5 Hecke forms

In this section we give some explicit Hecke forms for the lacunary pairs stated earlier. We will make use of [GS87], where the completions of all of these forms to eigenforms are found.

$(5,-1)$ and $(-1,5)$: These are forms of weight $k=2$, and $\epsilon=\epsilon_0=8$. The completion of $(5,-1)$ is $(5,-1)+2i\sqrt{2}(-1,5)$. This completion turns out to be an eigenform for all of the T_p with $p \nmid M = N\epsilon\epsilon_0 = 128$ (Here $\epsilon=\epsilon_0=8$). It is therefore lacunary by [Ser85]. To find the associated Hecke Character, we make use of 2.7. The quadratic field is $\mathcal{K}=\mathbb{Q}(\sqrt{-2})$, so that $|d|=8$. Thus $\mathcal{N}(\mathfrak{f}) \mid 16$ by 2.8, and we get $\mathfrak{f}=<4>$. Let $\mathfrak{a}$ be an ideal of $\mathfrak{O}_{\mathcal{K}}$ prime to $\mathfrak{f}$. Since $\mathcal{K}$ is a PID, $\mathfrak{a}=<\alpha>$, where $\alpha=a+b\sqrt{-2}\in\mathcal{K}$. In fact, we can normalize α so that $a\equiv 1 \pmod 4$. It turns out that $C(\mathfrak{a})=(-1)^{\frac{b(b-1)}{2}}\alpha$ is the Hecke Character, and one can easily check that it is independent of the choice of the generator α for the ideal $\mathfrak{a}$.

$(9,-3)$ and $(-3,9)$: These forms have weight $k=3$, and $e=e_0=8$. The field is $\mathcal{K}=\mathbb{Q}(i)$, and $\mathfrak{f}=(1+i)^5$ by the relation 2.7. The group of reduced residues mod $\mathfrak{f}$ is $<i><3><1+2i>$, and thus every $\mathfrak{a}=<\alpha>$ prime to $\mathfrak{f}$ can be written as $\alpha\equiv i^a3^b(1+2i)^c \pmod{(1+i)^5}$. It turns out that the Hecke character is $C(\mathfrak{a})=(-1)^{a+b}\alpha^2$.

$(3,-1)$ and $(-1,3)$: These forms have weight $k=1$, and $e=e_0=24$. Here $\mathcal{K}=\mathbb{Q}(i)$, the Gaussian field, and again making use of 2.7 we find that the conductor is $\mathfrak{f}=12(1+i)$ The group of reduced residues mod $4(1+i)$ in $\mathbb{Z}[i]$ is the direct product $<i><1+2i><1+4i>$, of order 16. The group of reduced residues mod 3 is $<1+2i>$, of order 8. Therefore if $\mathfrak{a}=<\alpha>$ is any ideal of

$\mathbf{Z}[i]$ prime to $\mathfrak{f} = 12(1+i)$, we have $\alpha \equiv i^a(1+2i)^b(1+4i)^c \pmod{4(1+i)}$ and $\alpha \equiv (1+2i)^d \pmod{3}$. The Hecke character is given by $C(\mathfrak{a}) = i^{2a+d}$.

(2, 2): here $k = 2, e = e_0 = 4$, and $\mathcal{K} = \mathbf{Q}(i)$. It is readily found that the conductor is $\mathfrak{f} =< 4 >$. Let $\mathfrak{a} =< \alpha >=< a + bi >$ be an ideal of $\mathfrak{O}_{\mathcal{K}} = \mathbf{Z}[i]$ prime to $\mathfrak{f}$, where α is normalized by setting $a \equiv 1 \pmod{4}$. Then the Hecke character is given by $C(\mathfrak{a}) = i^b$.

$(10, -4)$ and $(2, 4)$: These forms have weight $k = 3$, $e = 12$, and $e_0 = 3$. Here again $\mathcal{K} = \mathbf{Q}(i)$, the Gaussian field, and again making use of 2.7 we find that the conductor is $\mathfrak{f} = 3(1+i)$. The group of reduced residues mod $3(1+i)$ in $\mathbf{Z}[i]$ is $< 1 + 2i >$, of order 8. Therefore if $\mathfrak{a} =< \alpha >$ is any ideal of $\mathbf{Z}[i]$ prime to $\mathfrak{f} = 3(1+i)$, we have $\alpha \equiv (1+2i)^a \pmod{3(1+i)}$. There are two completions here, namely $F_{\pm}(\tau) := (10, -4) \pm 8(2, 4)$. If we let $C_+(\mathfrak{a}) = i^a\alpha^2$ and $C_-(\mathfrak{a}) = (-i)^a\alpha^2$, then $F_{\pm}(\tau) = H_{C_{\pm}}(\tau)$.

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