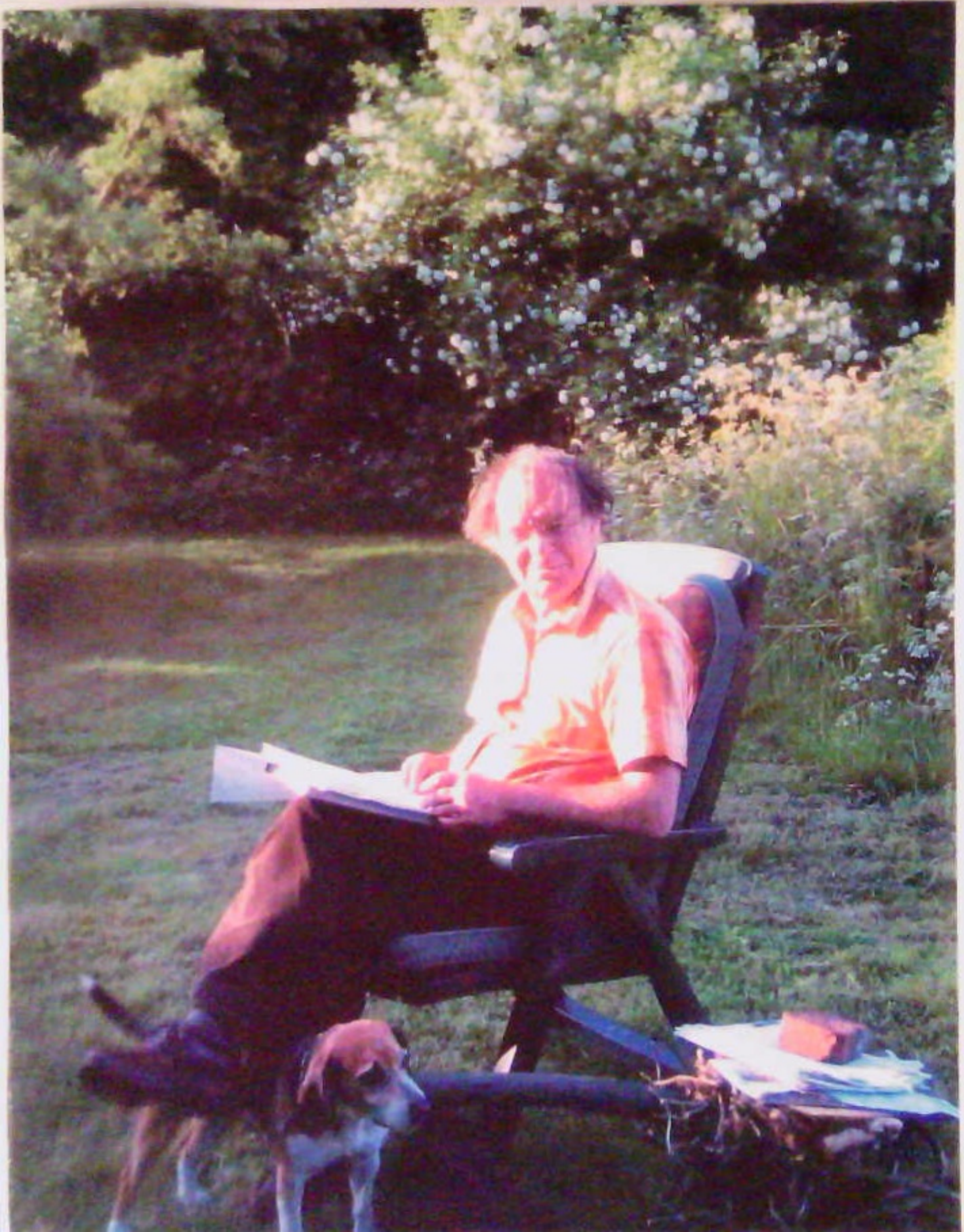
CONGRUENCES FOR THE RANK AND THE CRANK OF PARTITIONS

In memory of

Richard P. Lewis (1942–2007)

Frank Garvan
University of Florida



Richard P. Lewis died July 26 2007
aged 65.

1963 Queen's College Oxford
Algebraic Topology
Sir Michael Atiyah

1966 University of Sussex

1977 LMS

1980 Number Theory

1991 Sussex D Phil

2003 Retired
Open University MSc

"The generating functions of the rank and the crank modulo 8"
Ramanujan Journal, to appear

PARTITIONS

A *partition* of n is a nonincreasing sequences of positive integer whose sum is n.

$n=1$

(1) $\quad 1$

$n=2$

(2) $\quad 2$

$(1,1)$ $\quad 1+1$

$n=3$

(3) $\quad 3$

$(2,1)$ $\quad 2+1$

$(1,1,1)$ $\quad 1+1+1$

$n=4$

(4) $\quad 4$

$(3,1)$ $\quad 3+1$

$(2,2)$ $\quad 2+2$

$(2,1,1)$ $\quad 2+1+1$

$(1,1,1,1)$ $\quad 1+1+1+1$

②

Let $p(n) = \#$ of partitions of n

$$\sum_{n=0}^{\infty} p(n) q^n = 1 + q + 2q^2 + 3q^3 + 5q^4 + \cdots$$

$$= \prod_{n=1}^{\infty} \frac{1}{(1-q^n)}$$

$$= \frac{1}{1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + \cdots}$$

$$= 1 + \frac{q}{(1-q)^2} + \frac{q^4}{(1-q)^2(1-q^2)^2}$$

$$+ \frac{q^9}{(1-q)^2(1-q^2)^2(1-q^3)^2} + \cdots$$

$|q| < 1$

PARTITION CONGRUENCES

RAMANUJAN

$$p(5n+4) \equiv 0 \pmod{5}$$

$$p(7n+5) \equiv 0 \pmod{7}$$

$$p(11n+6) \equiv 0 \pmod{11}$$

ATKIN

$$p(11^3 \cdot 13 \cdot n + 237) \equiv 0 \pmod{13}$$

$$p(23^3 \cdot 17n + 2623) \equiv 0 \pmod{17}$$

$$p(13^3 \cdot 19n + 669) \equiv 0 \pmod{19}$$

$$p(5^4 \cdot 23n + 3474) \equiv 0 \pmod{23}$$

$$p(17^3 \cdot 29n + 31778) \equiv 0 \pmod{29}$$

$$p(103^3 \cdot 31n + 540228) \equiv 0 \pmod{31}$$

$$p(11^4 \cdot 13^3 \cdot 37n + 25315010) \equiv 0 \pmod{37}$$

$$\vdots$$

$$p(5^3 \cdot 7^4 \cdot 13^3 \cdot 17^3 \cdot 19^4 \cdot 37^4 \cdot 113 \cdot 337^3 \cdot 661^3 \cdot 1049^3 n + 1276827052061577611472274334763721410295) \equiv 0 \pmod{113}$$

④

ONO (2000)

Let $\ell \geq 5$ be prime & $m \geq 1$.
$\exists$ only (A, B) :

$$p(An + B) \equiv 0 \pmod{\ell^m}$$

DYSON'S RANK

The rank of a partition is the largest part minus the number of parts

The rank of $9+6+5+4+4+1$ is $9-6=3$

Dyson's rank explains Ramanujan's partition congruences mod 5 & 7

$$p(5n+4) \equiv 0 \pmod 5$$

$$p(7n+5) \equiv 0 \pmod 7$$

* The residue of the rank mod 5 divides the partitions of $5n+4$ into 5 equal classes.

Example

	Rank	(mod 5)
4	$4-1=3$	$\equiv 3$
3+1	$3-2=1$	$\equiv 1$
2+2	$2-2=0$	$\equiv 0$
2+1+1	$2-3=-1$	$\equiv 4$
1+1+1+1	$1-4=-3$	$\equiv 2$

CRANK (Andrews & G. 1988)

The crank of a partition is the largest part if there are no ones otherwise it is difference between the number of parts larger than the number of ones & the number of ones

The crank of

$$7+5+4+4+3+2+1+1+1+1$$

is $2-4=-2$.

The crank explains Ramanujans partition congruence mod 11

$$p(11n+6)\equiv 0 \pmod{11}$$

MAHLBURG (2005)

Let $M(r,t,n)$ = # of partitions of n with crank $\equiv r \pmod t$.

$$\sum_{r=0}^{t-1} M(r,t,n) = p(n)$$

Let $\ell \geq 5$ be prime.
Let $i, j \geq 1$. $\exists$ ∞ly many (A,B):

$$M(r, \ell^i, An+B) \equiv 0 \pmod{\ell^j}$$

for all $0 \leq r \leq \ell^i - 1$.

Corollary $p(An+B) \equiv 0 \pmod{\ell^j}$

BRINGMANN & ONO (preprint)

Let $N(r,t,n)$ = # of partitions of n with rank $\equiv r \pmod t$

$$\sum_{r=0}^{t-1} N(r,t,n) = p(n)$$

Let $\ell \geq 5$ be prime. Let $t \geq 1$ be odd such that $(\ell, t) = 1$. Let $j \geq 1$. $\exists$ ∞ ly (A,B):

$$N(r,\, t,\, An+B) \equiv 0 \pmod{\ell^j}$$

for all $0 \leq r \leq t-1$.

BRINGMANN (preprint)

Let $i \geq 1$.

$$N(r,\, \ell^i,\, An+B) \equiv 0 \pmod{\ell^j}$$

for all $0 \leq r \leq \ell^i - 1$.

COROLLARY $p(An+B) \equiv 0 \pmod{\ell^j}$

8A

Bringmann & Ono approach via weak Maas forms

Let $\ell > 3$ be prime.

$$R(z,q) = \sum_n \sum_m z^m N(m,n) q^n$$

Let $\zeta_\ell = \exp(2\pi i/\ell)$.

$$R(\zeta_\ell, q) = \sum_{r=0}^{\ell-1} \left(\sum_{n\geq 0} N(r,\ell,n) q^n \right) \zeta_\ell^r$$

Then for some integer μ, k

$$D(a,\ell;\tau) = q^\mu R(\zeta_\ell^a, q^k) + c \int_{-\bar\tau}^{i\infty} \frac{\Theta(a,\ell;kz)\,dz}{\sqrt{-i(\tau+z)}}$$

is a weak Maas form of weight $\frac{1}{2}$ on $\Gamma_1(144\ell^2)$.

Here $q = e^{2\pi i \tau}$ $(\mathrm{Im}\,\tau > 0)$

$$D(a,b;\tau) = \sum_{n=0}^{\infty}\left(\sum_{r=0}^{\ell-1} N(r,\ell,n)\,\zeta_\ell^{ra}\right) q^{kn+\nu} + c'\sum_{n} \gamma(a,b,y,n)\, q^{-k'n^2}$$

$$\left(q = e^{2\pi i \tau} = e^{2\pi i (x+iy)}\right)$$

<u>EXAMPLE</u> (G, preprint)

(A) $N(r, 11, 5^4 \cdot 11 \cdot 19^4 \cdot n + 4322599) \equiv 0 \pmod{11}$

$N(r, 11, 11^2 \cdot 19^4 \cdot n + 172904) \equiv 0 \pmod{11}$

for all $0 \leq r \leq 10$.

(B)

$N(r, 17, 7^4 \cdot 17^2 \cdot 41^4 \cdot n + 284663832)$

$\equiv N(0, 17, 17^2 \cdot 7^4 \cdot 41^4 \cdot n + 284663832) \pmod{17}$