RANK CONGRUENCES

ATKIN & HOSSAIN (1958)

$$2N(2,11,11n) + N(3,11,11n) + 7N(4,11,11n) + N(5,11,11n) \equiv 0 \pmod{11}$$

$$5N(1,11,11n+1) + 3N(2,11,11n+1) + 9N(3,11,11n+1) + 4N(4,11,11n+1) + N(5,11,11n+1) \equiv 0 \pmod{11}$$

$$\vdots$$

$$3N(0,11,11n+10) + 4N(1,11,11n+10) + 3N(4,11,11n+10) + N(5,11,11n+10) \equiv 0 \pmod{11}$$

(11)

O'BRIEN (1966)

$$3\,N(2,13,13n)+5\,N(3,13,13n)+$$
$$7\,N(4,13,13n)+10\,N(5,13,13n)$$
$$+\,N(6,13,13n)\equiv 0 \pmod{13}$$

$$\vdots$$

Fix $0 \leq r \leq 12$

LINEAR CONGRUENCE RELATIONS mod 13
FOR $N(k, 13, 13n+r)$ $0 \leq k \leq 6$

r	# of independent linear congruences
0	1
1	1
2	2
3	2
4	1
5	2
6	2
7	2
8	1
9	2
10	2
11	1
12	1

OBSERVATION:

# of inequiv. linear congruences mod 13 for $N(k, 13, 13n+r)$ $(0 \le k \le 6)$

$$= \begin{cases} 1 & \text{if } \left(\frac{1-24r}{13}\right) = +1 \\ 2 & \text{if } \left(\frac{1-24r}{13}\right) = 0 \text{ or } -1 \end{cases}$$

CRANK CONGRUENCES

For each prime t there is at least one linear congruence for $M(k, t, tn+r)$ $\left(0 \leq k \leq \frac{t}{2}\right)$ $(\bmod\ t)$ for each residue class mod t !!!!!

$$\sum_{k=1}^{n} k^2 M(k,n) = n\, p(n)$$

(DYSON(1989))

THERE ARE EXTRA CRANK CONGRUENCES MOD t for $t = 41, 53, 83$ and $120\,667\,369$

ATKIN & G. (2003)

LINEAR CONGRUENCES BETWEEN THE RANK AND THE CRANK

NONTRIVIAL

FOR EACH PRIME $t > 13$ THERE ARE FIVE LINEAR CONGRUENCES MOD t BEETWEEN THE RANK AND THE CRANK!!!

THESE COME FROM EXACT LINEAR RELATIONS BETWEEN RANK & CRANK MOMENTS

$$N_4(n) = -\left(2n + \frac{2}{3}\right) M_2(n) + \frac{8}{3} M_4(n) + (1 - 12n) N_2(n)$$

WHERE $N_{2k}(n) = \sum_m m^{2k} N(m,n)$ RANK MOMENT

$M_{2k}(n) = \sum_m m^{2k} M(m,n)$ CRANK MOMENT

THERE EXIST EXACT LINEAR RANK-CRANK MOMENT RELATIONS

$$\boxed{N_{2k}(n) = P_k(n)\, N_2(n) + \sum_{j=1}^{k} Q_{kj}(n)\, M_{2j}(n)}$$

FOR $k = 2, 3, 4$ and 5. HERE

$P_k(n) \in \mathbb{Z}[n]$ (degree $k-1$)

AND EACH

$Q_{k,j}(n) \in \mathbb{Q}[n]$ (degree $k-j$).

FOR $k=6$ THERE IS NO SUCH RELATION
FOR $k=7$ THERE IS A SIMILAR RELATION BUT INVOLVES AN EXTRA RANK MOMENT $N_{12}(n)$.

RANK-CRANK PDE (ATKIN & G. (2003))

$$z(q)_\infty^2 \left[C(z,q)\right]^3$$

$$= \left(3(1-z)^2\delta_q + \tfrac{1}{2}(1-z)^2\,\delta_z^2 - \tfrac{1}{2}(z^2-1)\,\delta_z + z\right) R(z,q)$$

WHERE

$$C(z,q) = \sum_{n\geq 0}\sum_{m} M(m,n)\, z^m q^n$$

$$R(z,q) = \sum_{n\geq 0}\sum_{m} N(m,n)\, z^m q^n$$

AND

$$\delta_q := q\frac{d}{dq}$$

$$\delta_z := z\frac{d}{dz}$$

CRANK MOMENTS ARE QUASI-MODULAR FORMS

Let $C_a(q) := \sum_{n\geq 1} M_a(n)\, q^n$ CRANK MOMENT GEN. FUNC.

Then

$$\boxed{C_a(q) = \delta_z^{\,a}\, C(z,q)\Big|_{z=1}}$$

$$C(z,q) = \prod_{n\geq 1} \frac{(1-q^n)}{(1-zq^n)(1-z^{-1}q^n)}$$

$$\delta_z\, C(z,q) = L(z,q)\, C(z,q)$$

WHERE

$$L(z,q) = \sum_{m,n\geq 1} \left(z^m q^{mn} - z^{-m} q^{mn}\right)$$

$$= \sum_{n\geq 1} \left(\frac{zq^n}{1-zq^n} - \frac{z^{-1}q^n}{1-z^{-1}q^n}\right)$$

WHEN j IS ODD

$$\delta_z^j L(z,q)\Big|_{z=1} = 2\sum_{m,n\geq 1} m^j q^{mn}$$

$$= 2\,\Phi_j(q)$$

WHERE

$$\Phi_j(q) = \sum_{n\geq 1} \sigma_j(n)\, q^n$$

$$\sigma_j(n) = \sum_{d|n} d^j$$

$$E_k(\tau) = 1 - \frac{2k}{B_k}\Phi_{k-1}(q)$$ EISENSTEIN SERIES

$$\boxed{\Phi_j = \frac{B_{j+1}}{2j+2}\left(1 - E_{j+1}(\tau)\right)}$$

$$\delta_z C = L\, C$$

$$\delta_z^a C = \sum_j \binom{a-1}{j} \delta_z^j(L)\, \delta_z^{a-1-j}(C)$$

$$\boxed{\begin{aligned} C_a &= 2\sum_{j=1}^{a/2-1} \binom{a-1}{2j-1} \Phi_{2j-1} C_{a-2j} \\ &\quad + 2\Phi_{a-1} P \\ \text{WHERE } P &= C(1,\xi) = \frac{1}{(\xi)_\infty} \end{aligned}}$$

COR.

$$C_{2k} = 2P \sum_{a_1+2a_2+\dots+ka_k=k} \alpha(a_1,\dots,a_k)\, \Phi_1^{a_1} \Phi_3^{a_2} \cdots \Phi_{2k-1}^{a_k}$$

$$= P \times \boxed{\text{SUM OF QUASI-MODULAR FORMS OF BOUNDED WEIGHT}}$$

RANK MOMENTS ARE ALMOST QUASI-MODULAR

BRINGMANN, G & MAHLBURG (SUBMITTED)

Define $R_{2k}(q) := \sum_{n \geq 1} N_{2k}(n) q^n$

RANK MOMENT GEN FUNC.

Then

$$R_{2k}(q) - P_k(\delta_q) R_2(q)$$

$$= P \times \boxed{\text{SUM OF QUASI-MODULAR FORMS OF BOUNDED WEIGHT}}$$

WHERE

$$P_k(x) = 2^{1-2k} \sum_{j=0}^{k-1} \binom{2k}{2j+1} (1-24x)^j$$

$$\in \mathbb{Z}[x]$$

If $\ell > 3$ is prime then

$$P_{\frac{\ell+1}{2}}(x) \equiv \frac{(\ell+1)}{2}\left(1 + (1-24x)^{\frac{\ell-1}{2}}\right) \pmod{\ell}$$

Let $U^{*}_{\varepsilon,\ell}$ be THE OPERATOR

$$U^{*}_{\varepsilon,\ell}\left(\sum a(n) q^n\right) = \sum_{\left(\frac{1-24n}{\ell}\right)=\varepsilon} a(n) q^n$$

for $\varepsilon \in \{-1, 0, 1\}$.

NOTE:

$$R_{\ell+1}(q) \equiv R_2(q) \pmod{\ell}$$

since $m^{\ell} \equiv m \pmod{\ell}$

$m^{\ell+1} \equiv m^2 \pmod{\ell}$

$$N_{\ell+1}(n) = \sum_m m^{\ell+1} N(m,n) \equiv N_2(n) \pmod{\ell}$$

$$\mathcal{U}^{*}_{\varepsilon,\ell}\left(R_{\ell+1} - P_{\frac{\ell+1}{2}}(\delta_6)R_2\right)$$

$$\equiv c_{\varepsilon,\ell}\, \mathcal{U}^{*}_{\varepsilon,\ell}(R_2) \pmod{\ell}$$

and $c_{\varepsilon,\ell} \not\equiv 0 \pmod{\ell}$
if $\varepsilon = 0$ or -1.

If $\varepsilon = 0$ or -1

$$U^{*}_{\varepsilon,\ell}(R_2)$$

$$\equiv U^{*}_{\varepsilon,\ell}(P\,G_\ell) \pmod{\ell}$$

WHERE G_ℓ is a sum of ℓ-integral quasimodular forms of BOUNDED WEIGHT.

THEOREM

Let $\ell > 3$ be PRIME.

β_ℓ be the such $24\beta_\ell \equiv 1 \pmod{\ell}$
$(1 \le \beta_\ell < \ell)$

$$r_\ell = \frac{24\beta_\ell - 1}{\ell}.$$

THEN

$$\sum_{n \geq 0} N_2(\ell n + \beta_\ell) q^{24n + r_\ell}$$

$$\equiv \eta^{r_\ell}(24\tau)\, G_\ell(24\tau) \pmod{\ell}$$

WHERE $G_\ell(\tau)$ is a sum of ℓ-integral ENTIRE MODULAR FORMS OF LEVEL 1 & BOUNDED WEIGHT.

SIMILAR CONGRUENCES EXIST FOR OTHER RANK MOMENTS

$N_{2k}(n) \quad (2 \le 2k < \ell - 1)$

WHEN $2k = \ell - 1$ THERE IS AN EXTRA TERM

EXAMPLE

Notation: Let $E(q) = \prod_{n=1}^{\infty}(1-q^n)$
$$= q^{-1/24}\,\eta(\tau)$$

$E_4(\tau)$, $E_6(\tau)$ be THE USUAL EISENSTEIN SERIES of WEIGHT 4 AND 6.

$$\sum_{n\geq 0} N_2(11n+6)q^n \equiv 3E^{13}(q) \pmod{11}$$

$$\sum_{n\geq 0} N_4(11n+6)q^n \equiv 7E^{13}(q) \pmod{11}$$

$$\sum_{n\geq 0} N_6(11n+6)q^n \equiv E^{13}(q)(4+E_4(\tau)) \pmod{11}$$

$$\sum_{n\geq 0} N_8(11n+6)q^n \equiv E^{13}(q)(5+6E_4(\tau)+6E_6(\tau)) \pmod{11}$$

$$\sum_{n\geq 0} N_{10}(11n+6)q^n \equiv E^{13}(q)(5+4E_4(\tau) + 6E_6(\tau) + 6E_4^2(\tau)) \pmod{11}$$

$$\sum_{n\geq 0} N(0,11,11n+6)\, q^n$$
$$\equiv E(q)^{13}\left(6 + 7E_4(\tau) + 5E_6(\tau) + 5E_4^2(\tau)\right) \pmod{11}$$

$$\sum_{n\geq 0} N(1,11,11n+6)\, q^n$$
$$\equiv E(q)^{13}\left(9 + 10E_6(\tau) + 5E_4^2(\tau)\right) \pmod{11}$$

$$\sum_{n\geq 0} N(2,11,11n+6)\, q^n$$
$$\equiv E(q)^{13}\left(4 + 3E_6(\tau) + 5E_4^2(\tau)\right) \pmod{11}$$

$$\sum_{n\geq 0} N(3,11,11n+6)\, q^n$$
$$\equiv E(q)^{13}\left(8 + 4E_4(\tau) + 6E_6(\tau) + 5E_4^2(\tau)\right) \pmod{11}$$

$\vdots$