The FORMS

$$\boxed{\begin{gathered}\eta^{13}(24\tau), \quad \eta^{13}(24\tau)E_4(24\tau) \\ \eta^{13}(24\tau)E_6(24\tau) \text{ AND } \eta^{13}(24\tau)E_4^2(24\tau)\end{gathered}}$$

ARE ALL HECKE EIGENFORMS
BUT OF DIFFERENT HALF-INTEGER WEIGHTS

SIMULTANEOUS RANK CONGRUENCES

$$N(k, 11, 5^4 \cdot 11 \cdot 19^4 \cdot n + 4322599) \equiv 0 \pmod{11}$$

and

$$N(k, 11, 11^2 \cdot 19^4 \cdot n + 172904) \equiv 0 \pmod{11}$$

for all $0 \leq k \leq 10$

THEOREM (G.)

Let $F(\tau) \in M_m(1)$ (Entire modular forms weight m & level 1)

Let $1 \leq r \leq 23$, and $(r,6)=1$.

Then
$G(\tau) = \eta^r(24\tau)\, F(24\tau)$ is a Hecke eigenform in $S_{\frac{r}{2}+m}(576, \chi_{12})$ if $\dim M_m(1) = 1$.

48 EXAMPLES

Let $G(\tau) = \sum_n a(n) q^n$

$$a(\ell^2 n) + \chi_{12}(\ell)\left(\frac{(-1)^k n}{\ell}\right)\ell^{k-1} a(n) + \ell^{2k-1} a\left(\frac{n}{\ell^2}\right)$$

$$= \lambda_\ell \, a(n)$$

ℓ prime, $\ell > 3$.

BORWEIN CONJECTURE (1990)

Let $n \geq 1$. Define polynomials $A_n(q)$, $B_n(q)$, $C_n(q)$ by

$$\prod_{j=1}^{n}(1-q^{3j-2})(1-q^{3j-1})$$
$$= A_n(q^3) - q\,B_n(q^3) - q^2\,C_n(q^3)$$

Then each of the polynomials $A_n(q)$, $B_n(q)$, $C_n(q)$ has nonnegative coefficients.

Example $(n=2)$

$$(1-q)(1-q^2)(1-q^4)(1-q^5)$$

$$= 1 - q - q^2 + q^3 - q^4 + 2q^6 - q^8 + q^9 - q^{10} - q^{11} + q^{12}$$

$$(1 + q^3 + 2q^6 + q^9 + q^{12})$$

$$- q(1 + q^3 + q^9)$$

$$- q^2(1 + q^6 + q^9)$$

$$A_2(q) = 1 + q + 2q^2 + q^3 + q^4$$

$$B_2(q) = 1 + q + q^3$$

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