RELATIONS BETWEEN THE RANKS AND CRANKS OF PARTITIONS

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See the Rankin Special ~~Issue of the~~ Ramanujan Journal
Vol 7 (2003), 343–366.

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Talk given at UIUC, April 10, 2003.

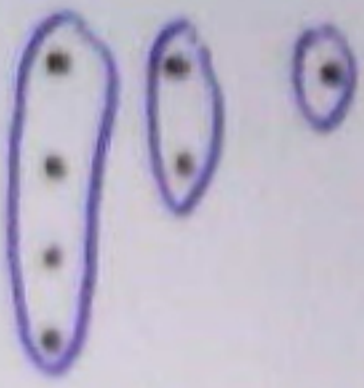
ABSTRACT. New identities and congruences involving the ranks and cranks of partitions are proved. The proof depends on a new PDE connecting their generating functions.

A **partition** of n is a representation of n is a sum of nonincreasing positive integers. Let $p(n)$ denote the number of partitions of n.

n	Partitions of n	$p(n)$
1	1	$p(1)=1$
2	2, 1+1	$p(2)=2$
3	3, 2+1, 1+1+1	$p(3)=3$
4	4, 3+1, 2+2, 2+1+1, 1+1+1+1	$p(4)=5$
5	5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1, 1+1+1+1+1	$p(5)=7$
6	6, 5+1, 4+2, 4+1+1, 3+3, 3+2+1, 3+1+1+1 2+2+2, 2+2+1+1, 2+1+1+1+1, 1+1+1+1+1+1	$p(6)=11$
7	7, 6+1, 5+2, 5+1+1, 4+3, 4+2+1, 4+1+1+1, 3+3+1, 3+2+2, 3+2+1+1 3+1+1+1+1 2+2+2+1, 2+2+1+1+1, 2+1+1+1+1+1, 1+1+1+1+1+1+1	$p(7)=15$

Graphical Rep. of a Partition

$3 + 2 + 1 + 1$



The conjugate partition is

$4 + 2 + 1$

GENERATION FUNCTIONS

$$\sum_{n\geq 0} p(n)\, q^{n} = \prod_{n=1}^{\infty} \frac{1}{1-q^{n}} = \frac{q^{\frac{1}{24}}}{\eta(\tau)}$$

$$= 1 + \frac{q}{(1-q)^2} + \frac{q^4}{(1-q)^2(1-q^2)^2} + \frac{q^9}{(1-q)^2(1-q^2)^2(1-q^3)^2} + \cdots$$

$$= 1 + \sum_{n\geq 1} \frac{q^{n^2}}{(1-q)^2(1-q^2)^2\cdots(1-q^n)^2}$$

$$\eta(\tau) = e^{\pi i \tau/12} \prod_{n\geq 1} \left(1 - e^{2\pi i n \tau}\right)$$

$$= q^{1/24} \prod_{n\geq 1} (1-q^n) \qquad \left(q = e^{2\pi i \tau}\right)$$

Ramanujan's Partition Congruences

$$p(5n+4) \equiv 0 \pmod{5}$$

$$p(7n+5) \equiv 0 \pmod{7}$$

$$p(11n+6) \equiv 0 \pmod{11}$$

Let $\alpha \geqslant 1$.

$$p(n) \equiv 0 \pmod{5^{\alpha}} \quad \text{if } 24n \equiv 1 \ (5^{\alpha})$$

$$p(n) \equiv 0 \pmod{7^{\lfloor \frac{\alpha}{2} \rfloor + 1}} \quad \text{if } 24n \equiv 1 \ (7^{\alpha})$$

$$p(n) \equiv 0 \pmod{11^{\alpha}} \quad \text{if } 24n \equiv 1 \ (11^{\alpha}).$$

Other partition congruences

Ono (2000), Ahlgren (2000)

Let $\ell \geq 5$ be prime, $m \geq 1$.

Then there exist A, B such that

$$p(An + B) \equiv 0 \pmod{\ell^m}$$

for all n.

Ahlgren & Boylan (2003)

Let p be prime.

If $p(pn+\delta) \equiv 0 \pmod{p}$ for all n, then

$p = 5, 7$ or 11.

DYSON's RANK of a partition is the largest part minus the number of parts.

Example The rank of the partition

$$3+3+2+1$$

is $3-4=-1$.

Let $N(m,n)$ denote the number of partitions of n with rank m.

Then

$$N(m,n) = N(-m,n).$$

Let $N(k,t,n)$ = number of partitions of n with rank $\equiv k \pmod{t}$.

Dyson (1944)
Atkin & Swinnerton-Dyer (1954)

$$N(0,5,5n+4) = N(1,5,5n+4) = N(2,5,5n+4)$$
$$= N(3,5,5n+4) = N(4,5,5n+4) = \frac{p(5n+4)}{5}$$

Example

Partitions of 4 RANK (mod 5)

4	$4-1\equiv 3$
$3+1$	$3-2\equiv 1$
$2+2$	$2-2\equiv 0$
$2+1+1$	$2-3\equiv -1\equiv 4$
$1+1+1+1$	$1-4\equiv -3\equiv 2$

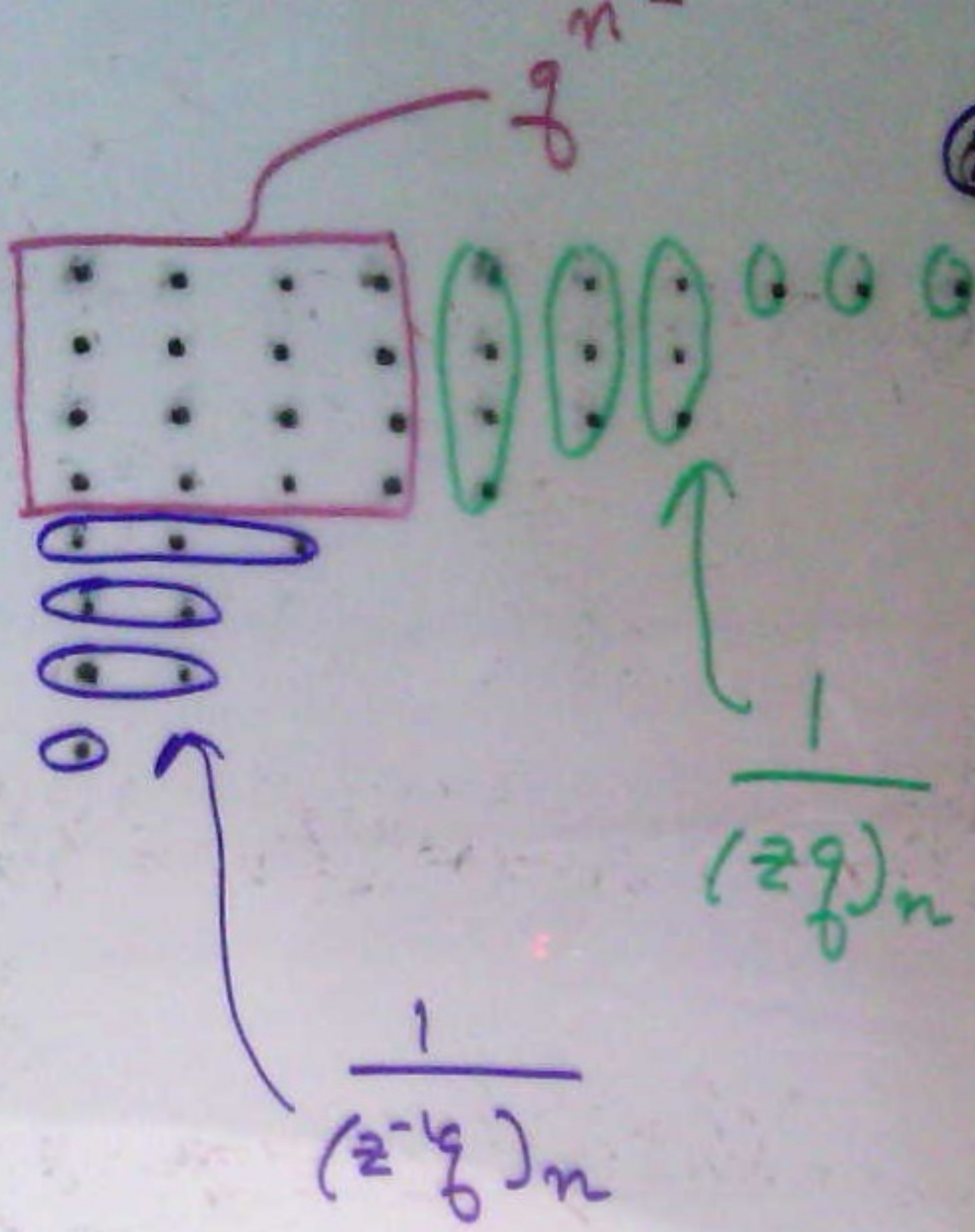
$$N(0,5,4)=N(1,5,4)=N(2,5,4)=N(3,5,4)$$
$$=N(4,5,4)=1$$

Also,

$$N(0,7,7n+5)=N(1,7,7n+5)=\cdots=N(6,7,7n+5)$$
$$=\frac{p(7n+5)}{7}.$$

Analog for partitions of $11n+6$ does NOT HOLD for rank mod 11.

DYSON CRANK CONJECTURE


q^{n^2}
$\frac{1}{(zq)_n}$
$\frac{1}{(z^{-1}q)_n}$

Generating function for the RANK

$$\boxed{R(z,q) := \sum_{n\geq 0} \sum_{m} N(m,n) z^m q^n}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(zq)_n (z^{-1}q)_n}$$

where

$$(a)_n = (a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}).$$

$$(a)_\infty = (a;q)_\infty = \lim_{n\to\infty} (a;q)_n \text{ if } |q|<1.$$

Also,

$$\boxed{\sum_{n\geq 0} N(m,n) q^n = \frac{1}{\prod_{n=1}^{\infty}(1-q^n)} \sum_{n=1}^{\infty} (-1)^{n-1} q^{\frac{n}{2}(3n-1)+|m|n} (1-q^n).}$$