Andrews & G. (1988) CRANK

of a partition is the largest part if the partition contains no ones otherwise it is the number of parts larger than the number of ones minus the number of ones.

Example

Let $\pi = 7+6+5+5+3+2+1+1+1+1+1$

$\mathrm{crank}(\pi) = 2-5 = -3.$

Let $M(m,n)$ denote the number of partitions of n with crank m.

Then

$$M(-m,n) = M(m,n)$$

for $n \neq 1$.

Let $M(k,t,n)$ denote the number of partitions of n with crank $\equiv k \pmod{t}$.

Then

$$M(0,11,11n+6) = M(1,11,11n+6)$$
$$= \cdots = M(10,11,11n+6) = \frac{p(11n+6)}{11}.$$

EXAMPLE

Partitions of 6	CRANK (mod 11)
6	$6 \equiv 6$
$5+1$	$1-1 \equiv 0$
$4+2$	$4 \equiv 4$
$4+1+1$	$1-2 \equiv 10$
$3+3$	$3 \equiv 3$
$3+2+1$	$2-1 \equiv 1$
$3+1+1+1$	$0-3 \equiv 8$
$2+2+2$	$2 \equiv 2$
$2+2+1+1$	$0-2 \equiv 9$
$2+1+1+1+1$	$0-4 \equiv 7$
$1+1+1+1+1+1$	$0-6 \equiv 5$

$M(0,11,6) = M(1,11,6) = \cdots = M(10,11,6) = 1$

Let $M(k,t,n)$ denote the number of partitions of n with crank $\equiv k \pmod t$.

Then

$$M(0,11,11n+6) = M(1,11,11n+6)$$
$$= \cdots = M(10,11,11n+6) = \frac{p(11n+6)}{11}.$$

<u>EXAMPLE</u>

Partitions of 6	CRANK (mod 11)
6	$6 \equiv 6$
$5+1$	$1-1 \equiv 0$
$4+2$	$4 \equiv 4$
$4+1+1$	$1-2 \equiv 10$
$3+3$	$3 \equiv 3$
$3+2+1$	$2-1 \equiv 1$
$3+1+1+1$	$0-3 \equiv 8$
$2+2+2$	$2 \equiv 2$
$2+2+1+1$	$0-2 \equiv 9$
$2+1+1+1+1$	$0-4 \equiv 7$
$1+1+1+1+1+1$	$0-6 \equiv 5$

$M(0,11,6) = M(1,11,6) = \cdots = M(10,11,6) = 1$

Generating function for CRANK

Let $M(m,n) = \#$ of partitions of n with crank m (for $n \neq 1$).

Define
$$M(0,1) = -1, \quad M(-1,1) = M(1,1) = 1,$$
$$M(m,1) = 0 \text{ if } m \neq 0, \pm 1.$$

Define

$$\boxed{C(z,q) := \sum_{n \geq 0} \sum_{m} M(m,n) z^m q^n}$$

$$= \prod_{n=1}^{\infty} \frac{(1-q^n)}{(1-zq^n)(1-z^{-1}q^n)}$$

Also,

$$\sum_{n \geq 0} M(m,n) q^n$$

$$= \frac{1}{\prod_{j=1}^{\infty}(1-q^j)} \sum_{n=1}^{\infty} (-1)^{n-1} q^{\frac{n}{2}(n-1)+|m|n} (1-q^n)$$

PDE

Define $\delta_z := z\frac{\partial}{\partial z}$ & $\delta_q := q\frac{\partial}{\partial q}$

Then

$$\boxed{\begin{aligned} &z(q)_\infty^2 \left[C(z,q) \right]^3 \\ &= \Big(3(1-z)^2 \delta_q + \tfrac{1}{2}(1-z)^2 \delta_z^{\,2} \\ &\qquad - \tfrac{1}{2}(z^2-1)\delta_z + z \Big) R(z,q) \end{aligned}}$$

where $(q)_\infty = \prod_{n=1}^{\infty}(1-q^n)$.

Define

$$R^*(z,q) = \frac{R(z,q)}{1-z}$$

$$C^*(z,q) = \frac{C(z,q)}{(1-z)}$$

PDE*

$$z(q)_\infty^2 \left[C^*(z,q) \right]^3$$

$$= \left(3\,\delta_q + \frac{1}{2}\,\delta_z + \frac{1}{2}\,\delta_z^2 \right) R^*(z,q)$$

LINEAR RANK-CRANK RELATIONS

$$N(1,5,5n+1) = N(2,5,5n+1)$$

Others for moduli 5, 7, 8, 9 and 12.

Atkin and Swinnerton-Dyer (1954)
Lewis (1991), (1992)
Santa-Gadea (1994)

$M(0,8,2n+1) + M(1,8,2n+1)$
$= M(3,8,2n+1) + M(4,8,2n+1)$

Others for moduli 5, 7, 8, 9, 10 and 11.

G. (1987), (1988), (1990)

$M(4,9,3n) = N(4,9,3n)$

Others for moduli 5, 7, 8 & 9.

G., Lewis, Santa-Gadea

DYSON

$$\sum_k \frac{k^2 N(k,n)}{p(n)}$$

$$= 2n - \frac{6}{\pi}\sqrt{\frac{3n}{2}} + O(1)$$

$$N(k,n) \sim M(k,n)$$

$$\sim \frac{1}{4}\frac{\pi}{\sqrt{6n}}\,\mathrm{sech}^2\left(\frac{\pi k}{2\sqrt{6n}}\right)$$

LINEAR RANK & CRANK MODULAR RELATIONS MOD p. (14)

$$2N(2,11,11n)+N(3,11,11n)$$
$$+7N(4,11,11n)+N(5,11,11n)\equiv 0 \pmod{11}$$

Other modular relations for the rank mod 11 and 13.
Atkin and Hussain (1958)
O'Brien (1966)

There is an analogous relation for the crank mod p for every prime p.

$$\boxed{\sum_k k^2 M(k,n) = 2n\,p(n)}$$

Dyson (1989)

PROOF:

$$C(z,q) = \sum_n \sum_k M(k,n) z^k q^n$$

$$= \prod_{n\geq 1} \frac{(1-q^n)}{(1-zq^n)(1-z^{-1}q^n)}$$

$$\left[\delta_z^2 C\right]_{z=1} = \sum_n \left(\sum_k k^2 M(k,n)\right) q^n$$

$$\delta_z C = \left(\sum_{n\geq 1} \frac{zq^n}{1-zq^n} - \frac{z^{-1}q^n}{1-z^{-1}q^n}\right) C$$

$$\delta_z^2 C = \left(\sum_{n\geq 1} \frac{zq^n}{(1-zq^n)^2} + \frac{z^{-1}q^n}{(1-z^{-1}q^n)^2}\right) C$$

$$+ \left(\sum_{n\geq 1} \frac{zq^n}{1-zq^n} - \frac{z^{-1}q^n}{1-z^{-1}q^n}\right) \delta_z C$$

$$\left[\delta_z^2 C\right]_{z=1} = \frac{2}{(q)_\infty} \sum_{n\geq 1} \frac{q^n}{(1-q^n)^2}$$

Let $P(q) = \frac{1}{(q)_\infty} = \sum_{n\geq 0} p(n) q^n$ (16)

$$\delta_q P = \left(\sum_{n\geq 1} \frac{n q^n}{1-q^n} \right) P$$

$$\sum_{n\geq 1} \boxed{n p(n)} q^n = \frac{1}{(q)_\infty} \sum_{n\geq 1} \frac{n q^n}{1-q^n}$$

$$= \frac{1}{(q)_\infty} \sum_{n\geq 1} \frac{q^n}{(1-q^n)^2}$$

$$= \sum_n \boxed{\left(\frac{1}{2} \sum_k k^2 M(k,n) \right)} q^n .$$

$\square$

There is an ~~extra~~ linear congruence for the crank mod p for $p = 41, 53, 83$ and 120667369.

For each prime $p > 13$ there are seven congruences involving both the rank and the crank mod p.

$$
\begin{aligned}
& 6N(0,29,29n+23) + 17N(1,29,29n+23) \\
& + 26N(2,29,29n+23) + 18N(3,29,29n+23) \\
& + 17N(4,29,29n+23) + 14N(5,29,29n+23) \\
& + 22N(6,29,29n+23) + 24N(7,29,29n+23) \\
& + 2N(9,29,29n+23) + 15N(10,29,29n+23) \\
& + 19N(11,29,29n+23) + 18N(12,29,29n+23) \\
& + 20N(13,29,29n+23) + 16N(14,29,29n+23) \\
\equiv\ & 11M(0,29,29n+23) + 17M(1,29,29n+23) \\
& + 28M(2,29,29n+23) + 26M(4,29,29n+23) \\
& + 6M(5,29,29n+23) + 28M(8,29,29n+23) \\
& \pmod{29}
\end{aligned}
$$

RANK & CRANK MOMENTS

For even $j \geq 2$ define

$$\boxed{\begin{aligned} N_j(n) &:= \sum_k k^j N(k,n) \\ M_j(n) &:= \sum_k k^j M(k,n) \end{aligned}}$$

$$\boxed{\begin{aligned} N_4(n) = {} & -(2n + 2/3)\, M_2(n) + \tfrac{8}{3} M_4(n) \\ & + (1-12n)\, N_2(n) \end{aligned}}$$

$$\boxed{\begin{aligned} N_6(n) = {} & \tfrac{2}{33}(324n^2+69n-10)\, M_2(n) \\ & + \tfrac{20}{33}(-45n+4)\, M_4(n) \\ & + \tfrac{18}{11} M_6(n) + (108n^2-24n+1)\, N_2(n) \end{aligned}}$$

1

(19)

For $k = 2, 3, 4, 5$ there are polynomials $P_k(n)$ of deg $k-1$ & $Q_{kj}(n)$ deg $k-j$ $(1 \le j \le k)$

such that

$$\boxed{N_{2k}(n) = P_k(n)\, N_2(n) + \sum_{j=1}^{k} Q_{kj}(n)\, M_{2j}(n)}$$

for $n \geq 0$.

$k=6$ NO RELATION

$k=7$ Similar relation with extra term $N_{12}(n)$.

PROOF OF PDE

Define $J(z,q) := (z)_\infty (z^{-1}q)_\infty$

$$= \prod_{n\geq 1} (1-zq^{n-1})(1-z^{-1}q^n).$$

$$S(z,\zeta,q) := \sum_{n=-\infty}^{\infty} \frac{(-1)^n \zeta^n q^{3n(n+1)/2}}{1-zq^n}$$

Then

A-SD (1954)

(*)
$$\boxed{\begin{aligned} &\zeta^3 S(z\zeta,\zeta^3,q) + S(z\zeta^{-1},\zeta^{-3},q) \\ &- \zeta \frac{J(\zeta^2,q)}{J(\zeta,q)} S(z,1,q) \\ &= \frac{J(\zeta,q)\,J(\zeta^2,q)\,(q)_\infty^2}{J(\zeta z,q)\,J(z,q)\,J(z\zeta^{-1},q)} \end{aligned}}$$

$$\boxed{C^*(z,q) = \frac{C(z,q)}{1-z} = \frac{(q)_\infty}{J(z,q)}}$$

G. (1988) 21

$$\boxed{\begin{aligned} R^*(z,q) &= R(z,q)/(1-z) \\ &= 1 + \frac{z}{(q)_\infty} S(z,1,q) \end{aligned}}$$

RHS of (*)

$$= 2\frac{(\zeta-1)^2 (q)_\infty^6}{[J(z,q)]^3} + \cdots$$

$$= 2(q)_\infty^3 \left[C^*[z,q)\right]^3 (\zeta-1)^2 + O((\zeta-1)^3)$$

RELATIONS BETWEEN RANK & CRANK MOMENTS

Recall,

$$N_j(n) = \sum_k k^j N(k,n)$$

$$M_j(n) = \sum_k k^j M(k,n)$$

Define

$$R_j = \sum_{n\geq 1} N_j(n) q^n$$

$$C_j = \sum_{n\geq 1} M_j(n) q^n$$

Then

$$\left[\delta_z^j R(z,q)\right]_{z=1} = \begin{cases} R_j & j \text{ even} \\ 0 & j \text{ odd} \end{cases}$$

$$\left[\delta_z^j C(z,q)\right]_{z=1} = \begin{cases} C_j & j \text{ even} \\ 0 & j \text{ odd} \end{cases}$$

Let a be even. Apply δ_z^a to both sides of the RANK-CRANK-PDE and set $z=1$:

$$\sum_{i=0}^{a/2-1}\binom{a}{2i}\sum_{\substack{\alpha+\beta+\gamma=a-2i\\ \alpha,\beta,\gamma\geq 0\\ \text{even}}}\binom{a-2i}{\alpha,\beta,\gamma}C_\alpha C_\beta C_\gamma P^{-2}$$

$$-\,3(2^{a-1}-1)\,C_2$$

$$=\frac{1}{2}(a-1)(a-2)\,R_a+6\sum_{i=1}^{a/2-1}\binom{a}{2i}(2^{2i-1}-1)\,\delta_q(R_{a-2i})$$

$$+\sum_{i=1}^{a/2-1}\left(\binom{a}{2i+2}(2^{2i+1}-1)-2^{2i}\binom{a}{2i+1}+\binom{a}{2i}\right)R_{a-2i}$$

Here

$$C_0=P=\frac{1}{(q)_\infty}$$

$\boxed{a=4}$

$$C_4 + 6\frac{C_2^2}{P} - C_2$$

$$= R_4 - R_2 + 12\,\delta_q(R_2)$$

Derivatives of Eisenstein Series

Following Ramanujan define

$$\Phi_j = \sum_{n=1}^{\infty} \frac{n^j q^n}{1-q^n} = \sum_{n=1}^{\infty} \sigma_j(n)\, q^n$$

$$E_n(\tau) = 1 - \frac{2n}{B_n}\Phi_{n-1}(q)$$

(n even)

$$E_2 = 1 - 24\,\Phi_1$$

$$E_4 = 1 + 240\,\Phi_3$$

$$E_6 = 1 - 540\,\Phi_5$$

For $n \geq 4$ (even), E_n is a modular form of weight n.

E_2 is a quasi ~~pseudo~~-modular form:

$$E_2(\tau+1) = E_2(\tau)$$

$$E_2\left(\frac{-1}{\tau}\right) = \tau^2 E_2(\tau) + \frac{6\tau}{\pi i}$$

$$\delta_q = q\frac{d}{dq}$$

Ramanujan

$$\delta_q(E_2) = (E_2^2 - E_4)/12$$

$$\delta_q(E_4) = (E_2 E_4 - E_6)/3$$

$$\delta_q(E_6) = (E_2 E_6 - E_4^2)/2$$

CRANK MOMENTS

$$C(z,q) = \prod_{n\geq 1} \frac{(1-q^n)}{(1-zq^n)(1-z^{-1}q^n)}$$

$$\boxed{\delta_z C(z,q) = L(z,q)\, C(z,q)}$$

where

$$L(z,q) = \sum_{n\geq 1} \left(\frac{zq^n}{1-zq^n} - \frac{z^{-1}q^n}{1-z^{-1}q^n} \right)$$

$$= \sum_{m,n\geq 1} \left(z^m q^{mn} - z^{-m} q^{mn} \right)$$

$$\delta_z^j L = \sum_{m,n\geq 1} \left(m^j z^m q^{mn} - (-m)^j z^{-m} q^{mn} \right)$$

$$\left(\delta_z^j L\right)_{z=1} = \begin{cases} 0 & \text{if } j \text{ even} \\ 2\Phi_j & \text{if } j \text{ odd} \end{cases}$$

Apply δ_z^{a-1} to both sides & set $z=1$:

$$\boxed{C_a = 2\sum_{j=1}^{\frac{a}{2}-1} \binom{a-1}{2j-1} \Phi_{2j-1} C_{a-2j}}$$

$$+ 2\,\Phi_{a-1}\, P$$

$$\boxed{C_2 = 2P\Phi_1}$$

$$\boxed{C_4 = 2P(\Phi_3 + 6\Phi_1^2)}$$

$$\boxed{\delta_q(C_2) = -\tfrac{P}{3}(6\Phi_1^2 - 5\Phi_3 - \Phi_1)}$$

$$\frac{6C_2^2}{P} = 12P\Phi_1^2$$

$$= \frac{5}{3}C_4 + \tfrac{1}{3}C_2 - 2\delta_q(C_2)$$

Hence

$$\tfrac{8}{3}C_4 - \tfrac{7}{3}C_2 - 2\,\delta_q(C_2)$$
$$= R_4 - R_2 + 12\,\delta_q(R_2)$$

So

$$\boxed{\begin{aligned} N_4(n) = &-\left(2n + \tfrac{2}{3}\right) M_2(n) + \tfrac{8}{3} M_4(n) \\ &+ (1 - 12n) N_2(n) \end{aligned}}$$

Quasi-Modular Forms

Let M_k be the space of modular forms of weight $2k$.

Then

$$\dim M_k = \begin{cases} \left[\frac{k}{6}\right] & \text{if } k \equiv 1 \ (6) \\ \left[\frac{k}{6}\right] + 1 & \text{otherwise} \end{cases}$$

Basis for M_k

$$= \left\{ E_4^{\,a} E_6^{\,b} : \ 2a + 3b = k, \ a, b \geq 0 \right\}$$

Let

$\boxed{\mathcal{W}_n}$ be the space spanned by the monomials $\Phi_1^{\,a} \Phi_3^{\,b} \Phi_5^{\,c}$

with $0 < a + 2b + 3c \leq n$.

Let $V_n = \mathrm{Span}\{\Phi_4^a \Phi_3^b \Phi_5^c : a+2b+3c=n\}$

Then
$$W_n = \sum_{k=1}^{n} V_k.$$

<u>Theorem</u> For $n \geq 1$

$$\dim V_n = \sum_{k=0}^{n} \dim M_k$$

$$\dim W_n = n + \sum_{k=2}^{n} (n-k+1) \dim M_k$$

k	$\dim M_k$	$\dim V_k$	$\dim W_k$
1	0	1	1
2	1	2	3
3	1	3	6
4	1	4	10
5	1	5	15
6	2	7	22
7	1	8	30
8	2	10	40
9	2	12	52
10	2	14	66

$$\boxed{\begin{array}{c} \delta_q(\mathcal{W}_n) \subset \mathcal{W}_{n+1} \\ \hline \delta_q^m(\mathcal{W}_n) \subset \mathcal{W}_{n+m} \end{array}}$$

$$\delta_q(P) = P\,\Phi_1$$

$$\delta_q^m(P) \in P\,\mathcal{W}_m$$

$$\delta_q^m(C_{2n}) \in P\,\mathcal{W}_{n+m}$$

Eg:

$$\delta_q(C_2) = -\tfrac{1}{3}P(6\Phi_1^2 - 5\Phi_3 - \Phi_1) \in P\,\mathcal{W}_2$$

$$\boxed{C_{2n} \in P\,\mathcal{W}_n}$$

Proof of Existence of Rank-Crank Moment relations

Let $a = 2k$ in $(*)$.

Let $k \geq 1$.

Let

$$T_k = (2k-1)(k-1)R_{2k} + 6\sum_{i=1}^{k-1}(\ )(\ )\delta_q(R_{2k-2i})$$

$$+ \sum_{i=1}^{k-1}\left[\binom{2k}{2i+2}(2^{2i+1}-1) - 2^{2i}\binom{2k}{2i+1} + \binom{2k}{2i}\right]R_{2k-2i}$$

$$T_k \in P\mathcal{W}_k$$

Let

$$\mathcal{C}_k = \{\delta_q^m(C_{2j}) : 1 \leq j \leq k,\ j+m \leq k\} \subset P\mathcal{W}_k$$

$$|\mathcal{C}_k| = \frac{k(k+1)}{2}$$

For $1 \le k \le 5$,

$$\dim W_k = \frac{k(k+1)}{2}.$$

$$\dim PW_k = \frac{k(k+1)}{2}.$$

Hence there must exist a linear relation between T_k & the elements of C_k.

$k=1$ $\quad T_1 = 0$ $\quad$ (trivial)

$2 \le k \le 5$ $\quad$ Relations

$k=6$ $\quad$ no relation

$k=7$ $\quad \dim W_7 = 30$

$\dim C_7 = 28$

$$T_6,\ \delta_1(T_6),\ T_7 \in PW_7$$

$$\boxed{\text{There must exist a rank-crank moment relation for } k=7.}$$

$$\Delta(q) = q \prod_{n=1}^{\infty} (1-q^n)^{24} \in \mathcal{M}_{12}$$

Define

$$\left[\sum_{n \geq 0} p_r(n) q^n = \prod_{n=1}^{\infty} (1-q^n)^r \right]$$

Then

$$P\Delta = \sum_{n \geq 1} p_{23}(n-1) q^n = q \prod_{n=1}^{\infty} (1-q^n)^{23} \in P\mathcal{W}_6$$

$$|C_6| = 21 \qquad \dim \mathcal{W}_6 = 22$$

There must exist a linear relation between

$P\Delta$, T_{12} & the functions in C_6.

35

$$P_{23}(n-1) =$$

$$= \frac{1}{8971966015628}\left(-57917897051578552 n^5 + \cdots + \right) M_2(n)$$

$$+ (\quad)(()n^4 + \cdots +) M_4(n)$$

$$+ (\quad)(()n^3 + \cdots +) M_6(n)$$

$$+ (\quad)(()n^2 + \cdots \quad) M_8(n)$$

$$+ (\quad)(-55565500 3092 n + \quad) M_{10}(n)$$

$$+ \frac{16986177}{1919176} M_{12}(n)$$

$$+ (\quad)(-466576 n^5 + \cdots) N_2(n)$$

$$- \frac{2459972 2121}{331633 6128} N_{12}(n)$$

Theorem For $n \geq 0$,

$$
\begin{aligned}
&4(n^2+n+14)(n^3+n^2+15)\, M_2(n) \\
&+(10n^4+2n^3+8n^2+21n+22)\, M_4(n) \\
&+13(n+14)(n^2+21n+13)\, M_6(n) \\
&+5n(n+6)\, M_8(n) + 15(n+19)\, M_{10}(n) \\
&+ M_{12}(n) + 12(n+10)(n+14)(n+19) \cdot (n+20)(n+21)\, N_2(n) \\
&+ N_{12}(n)
\end{aligned}
$$

$$
\equiv \begin{cases} (-1)^k \pmod{23} & \text{if } n = \dfrac{23k(3k\pm 1)}{2}+1 \\ 0 \pmod{23} & \text{otherwise} \end{cases}
$$