Lambert and Ramanujan

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Lambert-W, London, Ontario — Wednesday, July 27, 2016

ABSTRACT

We consider Lambert and generalized Lambert series studied by Ramanujan. We start with some weighted partition identities and divisor functions. Along the way we meet Ramanujan's mock theta functions and tau function. We show how we used MAPLE to discover, prove and check results. In particular we introduce a new MAPLE package, thetaids, for proving theta function identities. The results for the tau function are joint with Michael Schlosser.

LAMBERT (1728 – 1777)

RAMANUJAN (1887 – 1920)

Is there a q-analog of the Lambert W-function?

Notation

$$(a)_n = (a;q)_n = (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1})$$

$$(a)_\infty = (a;q)_\infty = \prod_{n=1}^{\infty}(1-aq^{n-1})$$

q-analog of $n!$

$$\begin{aligned}[n]_q &= 1\cdot(1+q)\cdot(1+q+q^2)\cdots(1+q+\cdots+q^{n-1})\\ &= \frac{(1-q)(1-q^2)\cdots(1-q^n)}{(1-q)^n}\\ &= \frac{(q;q)_n}{(1-q)^n}\end{aligned}$$

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$$e_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n} = \frac{1}{(z;q)_\infty} = \frac{1}{(1-z)(1-zq)(1-zq^2)\cdots}$$

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$$\lim_{q\to 1^-} e_q((1-q)z) = lim_{q\to 1^-} E_q((1-q)z) = \exp(z)$$

Let

$$w_q(z) = z + \sum_{n=2}^{\infty} a(n,q) \frac{z^n}{(q;q)_{n-1}}$$

such that

$$\widetilde{e}_q\left(w_q(z)\right) = z, \qquad \text{where } \widetilde{e}_q(z) = z\, e_q(-z)$$

Let

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Then

$$w_q(z) = z + \frac{z^2}{(1-q)} + \frac{q+2}{(1-q)(1-q^2)} z^3 + \frac{q^3+5q^2+5q+5}{(1-q)(1-q^2)(1-q^3)} z^4 + \cdot$$

```
> A:=n->subs(S,aa[n]):
> seq(degree(A(n),q),n=2..10);
                    0, 1, 3, 6, 10, 15, 21, 28, 36
>
> a:=(m,n)->coeff(A(n),q,m):
> seq(seq(a(m,n),m=0..(n-2)*(n-1)/2),n=2..5);
                 1, 2, 1, 5, 5, 5, 1, 14, 21, 31, 30, 19, 9, 1
```

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THE ON-LINE ENCYCLOPEDIA OF INTEGER SEQUENCES®

founded in 1964 by N. J. A. Sloane

1, 2, 1, 5, 5, 5, 1, 14, 21, 31, 30, 19, 9, 1 [Search] Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

Search: **seq:1,2,1,5,5,5,1,14,21,31,30,19,9,1**

Displaying 1-1 of 1 result found. page 1

Sort: relevance | references | number | modified | created Format: long | short | data

A152290 Coefficients in a q-analog of the LambertW function, as a triangle read by rows. +20 9

1, **1, 2, 1, 5, 5, 5, 1, 14, 21, 31, 30, 19, 9, 1**, 42, 84, 154, 210, 245, 217, 175, 105, 49, 14, 1, 132, 330, 708, 1176, 1722, 2148, 2386, 2358, 2080, 1618, 1086, 644, 294, 104, 20, 1, 429, 1287, 3135, 6006, 10164, 15093, 20496, 25188, 28770, 30225, 29511, 26571 (list; graph; refs; listen; history; text; internal format)

OFFSET 0,3

LINKS
Paul D. Hanna, Rows 0 to 30 of the triangle, flattened.
Eric Weisstein, q-Exponential Function from MathWorld.
Eric Weisstein, q-Factorial from MathWorld.

FORMULA
```
G.f.: A(x,q) = Sum_{n>=0} Sum_{k=0..n(n-1)/2} T(n,k)*q^k*x^n/faq(n,q),
  where faq(n,q) is the q-factorial of n.
G.f.: A(x,q) = (1/x)*Series_Reversion( x/e_q(x,q) ) where e_q(x,q) =
  Sum_{n>=0} x^n/faq(n,q) is the q-exponential function.
G.f. satisfies: A(x,q) = e_q( x*A(x,q), q) and A( x/e_q(x,q), q) =
  e_q(x,q).
```

⋮

KEYWORD	eigen,nonn,tabf
AUTHOR	Paul D. Hanna, Dec 02 2008
STATUS	approved

```
> seq(subs(q=0,A(n)),n=2..10);
                    1, 2, 5, 14, 42, 132, 429, 1430, 4862
> seq((subs(q=1,A(n))),n=2..10);
            1, 3, 16, 125, 1296, 16807, 262144, 4782969, 1000
> seq(subs(q=-1,A(n)),n=2..10);
                    1, 1, 4, 5, 36, 49, 512, 729, 10000
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$$\deg\ a(n+1,q) = \deg\ b(n+1,q) = \frac{n(n-1)}{2}$$

$$a(n+1,0) = C(n) = \frac{1}{n+1}\binom{2n}{n}$$

$$a(n+1,1) = (n+1)^{n-1}$$

$$a(n+1,-1) = (n+1)^{\lfloor (n-1)/2) \rfloor}$$

$$q^{n(n-1)/2} a\left(n+1, \frac{1}{q}\right) = b(n+1,q)$$

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LAMBERT SERIES

A **Lambert Series** has the form

$$S(q) = \sum_{n=1}^{\infty} a(n) \frac{q^n}{1-q^n}$$

Then

$$S(q) = \sum_{n=1}^{\infty} b(n) q^n,$$

where

$$b(n) = \sum_{d|n} a(d)$$

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EXAMPLES

$$\sum_{n=1}^{\infty} \mu(n) \frac{q^n}{1-q^n} = q, \qquad \sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\sum_{n=1}^{\infty} \phi(n) \frac{q^n}{1-q^n} = \frac{q}{(1-q)^2}, \qquad \sum_{d|n} \phi(d) = n$$

$$\sum_{n=1}^{\infty} n^k \frac{q^n}{1-q^n} = \sum_{n=1}^{\infty} \sigma_k(n) q^n, \qquad \sigma_k(n) = \sum_{d|n} d^k$$

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GENERALIZED LAMBERT SERIES

A **Generalized Lambert Series** has the form

$$S_k(z,\zeta,q)=\sum_{n=-\infty}^{\infty}\frac{\zeta^n q^{kn^2}}{(1-zq^n)}$$

JACOBI

$$2\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n(n+1)/2}}{1+q^n}=1+4\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n+1)/2}}{1+q^n}=\vartheta_4^2(q),$$

where

$$\vartheta_4(q)=\sum_{n=-\infty}^{\infty}(-1)q^{n^2}=\frac{(q;q)_\infty^2}{(q^2;q^2)_\infty}$$

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CARL JACOBI (1804 – 1851)

RAMANUJAN'S MOCK THETA FUNCTIONS

$$f(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2} = \frac{1}{(q;q)_\infty}\left(1 + 4\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}}{1+q^n}\right)$$

$$\omega(q) = \sum_{n\ge 0} \frac{q^{2n(n+1)}}{(q;q^2)_{n+1}^2} = g(q;q^2),$$

where

$$g(x,q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(x)_{n+1}(q/x)_{n+1}} = \frac{1}{(q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n-1)/2}}{1-xq^n}$$

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Mock Theta-Functions

Ramanujan (Jan. (1920))

$$(A)\quad 1+\frac{q}{(1-q)^2}+\frac{q^4}{(1-q)^2(1-q^2)^2}+\cdots$$

$$(B)\quad 1+\frac{q}{(1-q)}+\frac{q^4}{(1-q)(1-q^2)}+\cdots$$

Let $q=\exp(-t)$. Then

$$(A)=\sqrt{\frac{t}{2\pi}}\exp\left(\frac{\pi^2}{6t}-\frac{t}{24}\right)+o(1)$$

$$(B)=\sqrt{\frac{2}{5-\sqrt{5}}}\exp\left(\frac{\pi^2}{15t}-\frac{t}{60}\right)+o(1)$$

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DEFINITION (Ramanujan-Watson-Rhoades (1920 - 2013)) A **mock theta-function** is a function $f(q)$ of a complex variable q defined by an Eulerian form which converges for $|q| < 1$ and satisfies the following there conditions:

- infinitely many roots of unity are exponential singularities
- for every root of unity x there is a theta function $\vartheta_x(q)$ such that

$$f(q) - \vartheta_x(q) = O(1)$$

  as $q \to x$ (radially)
- f is not the sum of two functions one of which is a theta function and the other is a function that is radially bounded at all roots of unity

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Theorem (Ramanujan-Watson)

Let

$$f(q) = 1 + \frac{q}{(1+q)^2} + \frac{q^4}{(1+q)^2(1+q^2)^2} + \cdots$$

and

$$b(q) = \prod_{n=1}^{\infty} \frac{(1-q^n)^3}{(1-q^{2n})^2}.$$

Let ζ be a primitive k-th root of unity. Then as $q \to \zeta$ (radially)

- $f(q) = O(1)$ *if k is odd*
- $f(q) + b(q) = O(1)$ *if $k \equiv 2 \pmod 4$*
- $f(q) - b(q) = O(1)$ *if $k \equiv 0 \pmod 4$*

Folsom-Ono-Rhoades (2013), Zudilin (2013)

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GEORGE NEVILLE WATSON (1886 – 1965)

MOCK THETA TRANSFORMATION

$$q^{-1/24}f(q) = 2\sqrt{\frac{2\pi}{\alpha}}q_1^{4/3}\omega(q_1^2)+4\sqrt{\frac{3\alpha}{2\pi}}\int_0^{\infty} e^{-3\alpha x^2/2}\frac{\sinh \alpha x}{\sinh 3\alpha x/2}\,dx,$$

where

$$q = \exp(-\alpha), \quad q_1 = \exp(-\beta), \quad \alpha\beta = \pi^2.$$

$$q^{2/3}\omega(-q) = -\sqrt{\frac{\pi}{\alpha}}q_1^{2/3}\omega(-q_1) + \sqrt{\frac{12\alpha}{\pi}}\int_0^{\infty} e^{-3\alpha x^2}\frac{\sinh \alpha x}{\sinh 3\alpha x}\,dx$$

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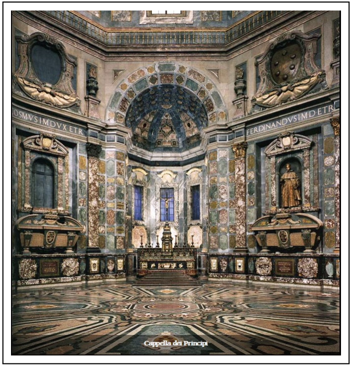
$$\int_0^\infty \exp(-3\pi x^2)\frac{\sinh(\pi x)}{\sinh 3\pi x}\,dx$$
$$= \frac{1}{\exp(2\pi/3)\sqrt{3}}\sum_{n=0}^{\infty}\frac{e^{-2n(n+1)\pi}}{(1+e^{-\pi})^2(1+e^{-3\pi})^2\cdots(1+e^{-(2n+1)\pi})^2}$$

...gives a thrill which is indistinguisable from the thrill I feel when I enter the Sagrestia Nuova of the Capelle Medicee ...



WEIGHTED PARTITION IDENTITIES

Let $p(n)$ denote the number of partitions of n.
Example The partitions of 4 are

$$4, \quad 3+1, \quad 2+2, \quad 2+1+1, \quad 1+1+1+1,$$

so that $p(4)=5$.

THEOREM [FOKKINK-FOKKINK-WANG (1995)]
Let $\mathcal{D}$ denote the set of partitions π into distinct parts. Then

$$\sum_{\substack{\pi\in\mathcal{D}\\ |\pi|=n}} (-1)^{\#(\pi)+1}\, s(\pi) = d(n), \tag{1}$$

where $\#(\pi)$ is the number of parts of π, $s(\pi)$ is the smallest part of π, $|\pi|$ is the sum of parts of π, and $d(n)$ is the number of divisors of n.

For $\pi \in \mathcal{D}$ define $ffw(\pi) = (-1)^{\#(\pi)+1}\, s(\pi)$.

EXAMPLE $n = 10$

π	$ffw(\pi)$
$[1,2,3,4]$	-1
$[2,3,5]$	2
$[1,4,5]$	1
$[1,3,6]$	1
$[4,6]$	-4
$[1,2,7]$	1
$[3,7]$	-3
$[2,8]$	-2
$[1,9]$	-1
$[10]$	10

$$\sum_{\substack{\pi\in\mathcal{D}\\ |\pi|=n}} ffw(\pi) = -1+2+1+1-4+1-3-2-1+10 = 4 = d(10)$$

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$$\sum_{\substack{\pi \in \mathcal{D} \\ |\pi| = n}} ffw(\pi) = -1+2+1+1-4+1-3-2-1+10 = 4 = d(10)$$

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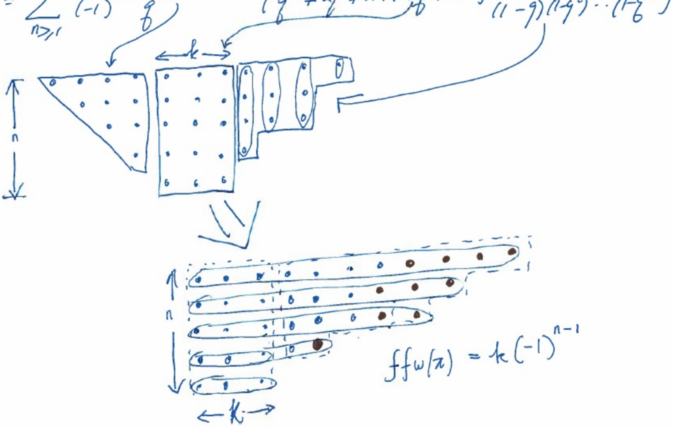
$$\sum_{n\geq 1} \frac{(-1)^{n-1} q^{n(n+1)/2}}{(q)_n (1-q^n)} = \sum_{n\geq 1} (-1)^{n-1} q^{n(n-1)/2} \cdot \frac{q^n}{(1-q^n)^2} \frac{1}{(q)_{n-1}}$$

$$= \sum_{n\geq 1} (-1)^{n-1} q^{1+2+\cdots+(n-1)} \left(q^n + 2q^{2n} + \cdots + kq^{kn} + \cdots\right) \frac{1}{(1-q)(1-q^2)\cdots(1-q^{n-1})}$$



ANDREWS PROOF

$$\sum_{n=1}^{\infty} \mathrm{FFW}(n) q^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^{n(n+1)/2}}{(q;q)_n (1-q^n)}$$

HEINE's q-ANALOG OF GAUSS's THEOREM

$$\sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n}{(c;q)_n (q;q)_n} \left(\frac{c}{ab}\right)^n = \frac{(c/a;q)_\infty (c/b;q)_\infty}{(c;q)_\infty (c/(ab);q)_\infty}$$

$$b^{-n}(b;q)_n = (b^{-1}-1)(b^{-1}-q)\cdots(b^{-1}-q^{n-1})$$

$$\Downarrow$$

$$(-1)^n q^{n(n-1)/2}$$

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In q-GAUSS let $a = z$, $c = zq$ and $b \to \infty$:

$$\sum_{n=0}^{\infty} \frac{(z;q)_n(-1)^n q^{n(n+1/2}}{(zq;q)_n(q;q)_n} = \frac{(q;q)_\infty}{(zq;q)_\infty}$$

$$\left[\frac{d}{dz}\frac{(z;q)_n}{(zq;q)_n}\right]_{z=1} = \left[\frac{d}{dz}(1-z)\frac{(zq;q)_{n-1}}{(zq;q)_n}\right]_{z=1} = -\frac{1}{1-q^n}.$$

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THEOREM [G. 2016]

Let $\mathcal{P}_o$ be the set of partitions in which all parts except possibly the largest part are odd and all odd positive integers less than or equal to the largest part occur as parts. Then

$$\sum_{\substack{\pi\in\mathcal{P}_o\\ |\pi|=n}} (-1)^{(\ell_O(\pi)-1)/2} = d_1(n),$$

where $d_1(n)$ is the number of odd divisors of n.

$$\sum_{\substack{\pi\in\mathcal{P}_o\\ |\pi|=n}} (-1)^{(\ell_O(\pi)-1)/2+\#(\pi)+1} = (-1)^{n(n-1)/2}\, d_{8,1}(n),$$

where $d_{8,1}(n)$ is the number of divisors of n congruent to $\pm1 \pmod 8$ minus the number of divisors of n congruent to $\pm3 \pmod 8$. Here $\ell_O(\pi)$ is the largest odd part and $\#(\pi)$ is the number of parts of π.

DEFINE

$$\omega_{1a}(\pi) = (-1)^{(\ell_O(\pi)-1)/2},$$
$$\omega_{1b}(\pi) = (-1)^{(\ell_O(\pi)-1)/2+\#(\pi)+1}.$$

EXAMPLE $n = 9$

π	$\ell_O(\pi)$	$\#(\pi)$	$\omega_{1a}(\pi)$	$\omega_{1b}(\pi)$
1^9	1	9	1	1
$1^7 2^1$	1	8	1	-1
$1^5 2^2$	1	7	1	1
$1^3 2^3$	1	6	1	-1
$1 2^4$	1	5	1	1
$1^6 3^1$	3	7	-1	-1
$1^3 3^2$	3	5	-1	-1
$1^2 3^1 4^1$	3	4	-1	1
$1^1 3^1 5^1$	5	3	1	1

$$\sum_{\substack{\pi\in\mathcal{P}_o \\ |\pi|=9}} \omega_{1a}(\pi) = 3 = d_1(9), \qquad \sum_{\substack{\pi\in\mathcal{P}_o \\ |\pi|=9}} \omega_{1b}(\pi) = 1 = (-1)^{36} d_{8,1}(9).$$

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LOVEJOY (2004)

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$$\sum_{n=1}^{\infty} \frac{q^{n^2}}{(-q;q^2)_n (1+q^{2n})} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^{n(n+1)/2} (q;q)_{n-1}}{(-q;q)_n}$$

$$= \sum_{n=1}^{\infty} \sum_{j=-n+1}^{n} (-1)^{n+j} q^{2n^2-j^2} = \sum_{n=1}^{\infty} (-1)^{n(n-1)/2} d_{8,1}(n) q^n.$$

LOVEJOY (2004)

MAPLE `thetaids` PACKAGE

The building block of JACOBI's theta functions is the triple-product:

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The DEDEKIND ETA-FUNCTION

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)$$

where $q = e^{2\pi i \tau}$.
The GENERALIZED DEDEKIND ETA FUNCTION

$$\eta_{\delta,g}(\tau) = q^{\frac{\delta}{2} P_2(g/\delta)} \prod_{m \equiv \pm g \pmod{\delta}} (1 - q^m),$$

where $P_2(t) = \{t\}^2 - \{t\} + \frac{1}{6}$ is the second periodic Bernoulli polynomial, $\{t\} = t - [t]$ is the fractional part of t, $g, \delta, m \in \mathbb{Z}^+$ and $0 \leq g < \delta$.

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GENERALIZED DEDEKIND ETA-PRODUCT OF LEVEL N

$$f(\tau) = \prod_{\substack{\delta \mid N \\ 0<g<\delta}} \eta_{\delta,g}^{r_{\delta,g}}(\tau),$$

where $r_{\delta,g} \in \mathbb{Z}$.

$$\eta_{N,g_1}(\tau)^{r_1}\, \eta_{N,g_1}(\tau)^{r_1} \cdots \eta_{N,g_m}(\tau)^{r_m}$$

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THEOREM [Robins, 1991]

The function $f(\tau)$ is a modular function on $\Gamma_1(N)$ if

(i) $$\sum_{\substack{\delta | N \\ g}} \delta P_2(\tfrac{g}{\delta}) r_{\delta,g} \equiv 0 \pmod 2, \text{ and}$$

(ii) $$\sum_{\substack{\delta | N \\ g}} \frac{N}{\delta} P_2(0) r_{\delta,g} \equiv 0 \pmod 2.$$

THE VALENCE FORMULA

Let $f \neq 0$ be a modular form of weight k with respect to a subgroup Γ of finite index in $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$. Then

$$\mathrm{ORD}(f,\Gamma) = \frac{1}{12}\mu\, k,$$

where μ is index $\widehat{\Gamma}$ in $\widehat{\Gamma(1)}$,

$$\mathrm{ORD}(f,\Gamma) := \sum_{\zeta\in R^*} \mathrm{ORD}(f,\zeta,\Gamma),$$

R^* is a fundamental region for Γ, and

$$\mathrm{ORD}(f,\zeta,\Gamma) = \kappa(\zeta,\Gamma)\,\mathrm{ord}(f,\zeta),$$

for a cusp ζ and $\kappa(\zeta,\Gamma)$ denotes the fan width of the cusp ζ (mod Γ).

TO PROVE AN ALLEGED THETA-FUNCTION IDENTITY FIRST WE WRITE IT IN THE FORM

$$\alpha_1 f_1(\tau) + \alpha_2 f_2(\tau) + \cdots + \alpha_n f_n(\tau) + 1 = 0,$$

where each $\alpha_i \in \mathbb{C}$ and each $f_i(\tau)$ is a generalized eta-product of level N.

ALGORITHM:

STEP 1. Check each $f_j(\tau)$ is a generalized eta-product on $\Gamma_1(N)$.

STEP 2. Find a set $\mathcal{S}_N$ of inequivalent cusps for $\Gamma_1(N)$ and the fan width of each cusp.

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$$B = \sum_{\substack{s \in \mathcal{S}_N \\ s \neq i\infty}} \min(\{\mathrm{ORD}(f_j, s, \Gamma_1(N)) \,:\, 1 \leq j \leq n\} \cup \{0\}).$$

STEP 5. Show that

$$\mathrm{ORD}(g(\tau), i\infty, \Gamma_1(N)) > -B$$

where

$$g(\tau) = 1 + \alpha_1 f_1(\tau) + \alpha_2 f_2(\tau) + \cdots + \alpha_n f_n(\tau).$$

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ROGERS-RAMANUJAN FUNCTIONS

The Rogers-Ramanujan functions are

$$G(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})},$$

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RAMANUJAN FORTY IDENTITIES

$$H(q)G(q^{11}) - q^2 G(q)H(q^{11}) = 1.$$

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THE ROGERS-RAMANUJAN CONTINUED FRACTION

$$R(q) = \frac{q^{\frac{11}{60}} H(q)}{q^{-\frac{1}{60}} G(q)} = \frac{\eta_{5,1}(\tau)}{\eta_{5,2}(\tau)} = q^{\frac{1}{5}} \prod_{n=1}^{\infty} \frac{(1-q^{5n-1})(1-q^{5n-4})}{(1-q^{5n-2})(1-q^{5n-3})}$$

$$= \cfrac{q^{1/5}}{1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{1+\ddots}}}}$$

PROOF OF $H(q)G(q^{11}) - q^2G(q)H(q^{11}) = 1$

```
> with(thetaids):
> with(qseries):
> G:=j->1/GetaL([1],5,j):H:=j->1/GetaL([2],5,j):
> G(1), H(1);
```

$$\frac{JAC(0,5,\infty)}{JAC(1,5,\infty)}q^{-\frac{1}{60}},\quad \frac{JAC(0,5,\infty)}{JAC(2,5,\infty)}q^{\frac{11}{60}}$$

```
> jac2getaprod(G(1)), jac2getaprod(H(1));
```

$$\frac{1}{\eta_{5,1}(\tau)},\quad \frac{1}{\eta_{5,2}(\tau)}$$

```
> JACID:= H(1)*G(11) - G(1)*H(11)-1;
```

$$\frac{JAC(0,5,\infty)\,JAC(0,55,\infty)}{JAC(2,5,\infty)\,JAC(11,55,\infty)}-\frac{q^2\,JAC(0,5,\infty)\,JAC(0,55,\infty)}{JAC(1,5,\infty)\,JAC(22,55,\infty)}-1$$

```
> series(jac2series(%,1000),q,1000);
```

$$O\left(q^{1000}\right)$$

```
> provemodfuncidBATCH(JACID,55);
*** There were NO errors.  Each term was modular
function on Gamma1(55).  Also -mintotord=40.
To prove the identity we need to check up to
O(q^(42)).
To be on the safe side we check up to O(q^(150)).
*** The identity is PROVED!
1
```

ROBINS GENERALIZED ETA IDENTITIES

$$G(3)\,H(1) - G(1)\,H(3) = 1,$$

where

$$G(n) = \frac{1}{\eta_{13,1}(n\tau)\eta_{13,3}(n\tau)\eta_{13,4}(n\tau)},$$

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GENERALIZATIONS OF ROBINS GENERALIZED ETA PRODUCTS

DEFINE

$$G(n, N, \chi) = G(n) := \prod_{\substack{\chi(g)=1 \\ 0<g<\frac{N}{2}}} \frac{1}{\eta_{N,g}(n\tau)},$$

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where χ is non-principal real Dirichlet character mod N satisfying $\chi(-1) = 1$.
ROBINS: $N = 5, 8, 12, 13$ (14 identities)

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JIE FRYE and G. (2013–2016): $N = 5, 8, 10, 12, 13, 15, 17, 21, 24, 26, 28, 30, 34, 40, 42, 56, 60$ (134 identities)

$$G(1) = G(1, 17, \left(\frac{\cdot}{17}\right)) = \frac{1}{\eta_{17,1}(\tau)\,\eta_{17,2}(\tau)\,\eta_{17,4}(\tau)\,\eta_{17,8}(\tau)}$$
$$= \frac{q^{-2/3}}{(q, q^2, q^4, q^8, q^9, q^{13}, q^{15}, q^{16}; q^{17})_\infty},$$

$$H(1) = H(1, 17, \left(\frac{\cdot}{17}\right)) = \frac{1}{\eta_{17,3}(\tau)\,\eta_{17,5}(\tau)\,\eta_{17,6}(\tau)\,\eta_{17,7}(\tau)}$$
$$= \frac{q^{4/3}}{(q^3, q^5, q^6, q^7, q^{10}, q^{11}, q^{12}, q^{14}, ; q^{17})_\infty}.$$

$N = 17$

$$G(2)\,H(1) - G(1)\,H(2) = 1, \quad \Gamma_1(34), \quad -B = 16.$$

We consider 10 types of identities. For example:

TYPE 3

Assume N and χ are fixed. We consider identities of the form

$$\frac{G(a_1)\,G(b_1) \pm H(a_1)\,H(b_1)}{G(a_2)\,H(b_2) \pm H(a_2)\,H(b_2)} = f(\tau),$$

which are not a quotient of Type 1 and 2 identities, and where $f(\tau)$ is an eta-product, a_1, b_1 ,a_2, b_2 are positive relatively prime integers, $a_1 b_1 = a_2 b_2$.

`findtype3(T)` — cycles through symbolic expressions

$$\frac{\_G(a_1)\,\_G(b_1) + c_1\,\_H(a_1)\,\_H(b_1)}{\_G(a_2)\,\_H(b_2) + c_2\,\_H(a_2)\,\_G(b_2)}$$

where $2 \le n \le T$, $a_1 b_1 = a_2 b_2 = n$, $(a_1, b_1, c_1, d_1) = 1$, $a_1 \le b_1$, $b_2 < a_2$, $c_1, c_2 \in \{-1, 1\}$, the following are integers

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$$\frac{G(a_1)\,G(b_1) \pm H(a_1)\,H(b_1)}{G(a_2)\,H(b_2) \pm H(a_2)\,H(b_2)} = f(\tau),$$

which are not a quotient of Type 1 and 2 identities, and where $f(\tau)$ is an eta-product, a_1, b_1 ,a_2, b_2 are positive relatively prime integers, $a_1 b_1 = a_2 b_2$.

`findtype3(T)` — cycles through symbolic expressions

$$\frac{\_G(a_1)\,\_G(b_1) + c_1\,\_H(a_1)\,\_H(b_1)}{\_G(a_2)\,\_H(b_2) + c_2\,\_H(a_2)\,\_G(b_2)}$$

where $2 \le n \le T$, $a_1 b_1 = a_2 b_2 = n$, $(a_1, b_1, c_1, d_1) = 1$, $a_1 \le b_1$, $b_2 < a_2$, $c_1, c_2 \in \{-1, 1\}$, the following are integers

$$\mathtt{GE}(a_1){+}\mathtt{GE}(b_1){-}(\mathtt{HE}(a_2){+}\mathtt{HE}(b_2)), \quad \mathtt{GE}(a_2){+}\mathtt{HE}(b_2){-}(\mathtt{HE}(a_2){+}\mathtt{GE}(b_2)),$$

We consider 10 types of identities. For example:

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$\mathtt{GE}(a_1)+\mathtt{GE}(b_1)-(\mathtt{HE}(a_2)+\mathtt{HE}(b_2))$, $\mathtt{GE}(a_2)+\mathtt{HE}(b_2)-(\mathtt{HE}(a_2)+\mathtt{GE}(b_2))$, and $[a_2, b_2, c_2]$ is not an element for the list `myramtype1` (product earlier by `findtype1`), using `CHECKRAMIDF` to check whether the expression corresponds to alikely eta-product, and if so uses `provemodfuncidBATCH` to prove it. The procedure also returns a list of $[a_1, b_1, c_1, a_2, b_2, c_2]$ which give identities. Here

$$\mathrm{GE}(a) = \mathrm{ord}(G(a), i\infty), \qquad \mathrm{HE}(a) = \mathrm{ord}(H(a), i\infty).$$

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```
> G:=j->1/GetaL([1,3,4],13,j):
> H:=j->1/GetaL([2,5,6],13,j):
> GE:=j->-GetaLEXP([1,3,4],13,j):
> HE:=j->-GetaLEXP([2,5,6],13,j):
> findtype3(160);
*** There were NO errors.  Each term was modular
function on Gamma1(26).  Also -mintotord=18.
To prove the identity we need to check up to
O(q^(20)).  To be on the safe side we
check up to O(q^(70)).
*** The identity below is PROVED!
```

$[1, 2, 1, 2, 1, -1]$

$$\frac{G(1)\,G(2) + H(1)\,H(2)}{G(2)\,G(1) - H(2)\,H(1)} = \frac{\eta(2\tau)^2\eta(13\tau)^2}{\eta(\tau)^2\eta(26\tau)^2}$$

```
*** There were NO errors.  Each term was modular
function on Gamma1(130).  Also -mintotord=432.
To prove the identity we need to check up to
O(q^(434)).
To be on the safe side we check up to O(q^(692)).
*** The identity below is PROVED!
```

$[2, 5, 1, 10, 1, -1]$

$$\frac{G(2)\,G(5) + H(2)\,H(5)}{G(10)\,G(1) - H(10)\,H(1)} = 1$$

```
*** There were NO errors.  Each term was modular
function on Gamma1(182).  Also -mintotord=864.
To prove the identity we need to check up to
O(q^(866)).
To be on the safe side we check up to O(q^(1228)).
*** The identity below is PROVED!
```

$[1, 14, 1, 7, 2, -1]$

$$\frac{G(1)\,G(14)+H(1)\,H(14)}{G(7)\,G(2)-H(7)\,H(2)}=\frac{\eta(2\tau)\eta(7\tau)\eta(26\tau)\eta(91\tau)}{\eta(\tau)\eta(13\tau)\eta(14\tau)\eta(182\tau)}$$

```
"n=", 50
"n=", 100
"n=", 150
```

$$[[1, 2, 1, 2, 1, -1], [2, 5, 1, 10, 1, -1], [1, 14, 1, 7, 2, -1]]$$

RAMANUJAN'S TAU FUNCTION

Ramanujan's tau function $\tau(n)$:

$$\sum_{n\geq 1} \tau(n)\, q^n = q(q;q)_\infty^{24} = q\prod_{n=1}^{\infty}(1-q^n)^{24}$$

- $\tau(n)$ is multiplicative; $\tau(mn) = \tau(m)\,\tau(n)$ for $(m,n) = 1$.
- $\tau(p^{r+1}) = \tau(p)\tau(p^r) - p^{11}\tau(p^{r-1})$ for p prime and $r > 0$.
- $|\tau(p)| \leq 2p^{11/2}$ for all primes p.
- Lehmer (1947) conjectured that $\tau(n) \neq 0$ for all n

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FORMULAS FOR TAU

RAMANUJAN

$$\tau(n) = \frac{65}{756}\sigma_{11}(n) + \frac{691}{756}\sigma_5(n) - \frac{691}{3}\sum_{m=1}^{n-1}\sigma_5(m)\sigma_5(n-m).$$

DYSON (1972)

$$\tau(n) = \sum \frac{(a-b)(a-c)(a-d)(a-e)(b-c)(b-d)(b-e)(c-d)(c-e)(d-e)}{1!2!3!4!}$$

where the sum is over integers a, b, c, d, e satisfying

$$a, b, c, d, e \equiv 1, 2, 3, 4, 5 \pmod 5$$
$$a + b + c + d + e = 0$$
$$a^2 + b^2 + c^2 + d^2 + e^2 = 10n$$

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A PARTITION FORMULA FOR TAU

Recall Ramanujan's identity

$$H(q)G(q^{11}) - q^2 G(q)H(q^{11}) = 1,$$

where

$$G(q) = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})},$$

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A SHIFTED PARTITION IDENTITY

Let

$$\alpha(n) = \text{the number of partitions of } n \text{ with parts congruent to } \pm 1 \pmod 5, \quad \pm 22 \pmod{55},$$

$$\beta(n) = \text{the number of partitions of } n \text{ with parts congruent to } \pm 2 \pmod 5, \quad \pm 11 \pmod{55}.$$

THEN

$$\alpha(n) = \beta(n-2),$$

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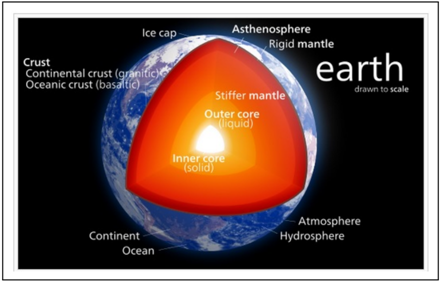
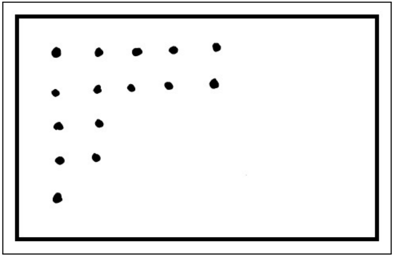
p-CORES


Ice cap
Asthenosphere
Rigid mantle
earth
drawn to scale
Crust
Continental crust (granitic)
Oceanic crust (basaltic)
Stiffer mantle
Outer core
(liquid)
Inner core
(solid)
Atmosphere
Hydrosphere
Continent
Ocean

p-CORES



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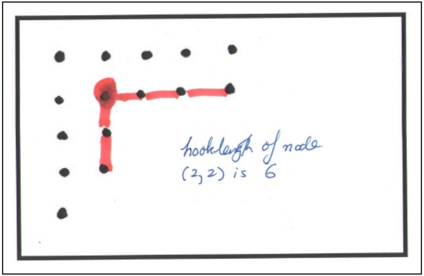

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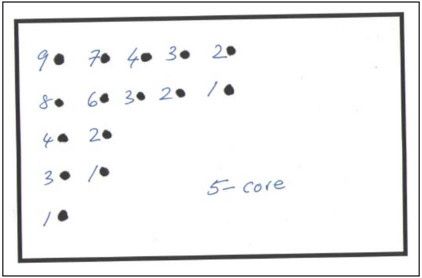

A partition is a **p-core** if it has no hook-lengths that are multiples of p. Let $\mathcal{T}_p$ be the set of p-cores. So

$$(5,5,2,2,1) \in \mathcal{T}_5$$

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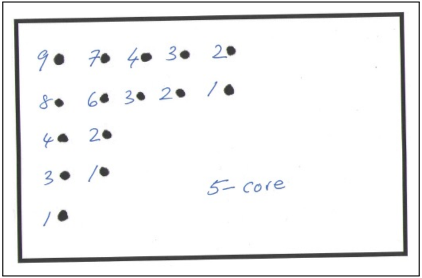

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p-CORES

Let $a_p(n)$ be the number of partitions of n that are p-cores.
THEN

$$\sum_{n=0}^{\infty} a_p(n)\, q^n = \prod_{n=1}^{\infty} \frac{(1-q^{pn})^p}{(1-q^n)} = \frac{(q^p;q^p)_\infty^p}{(q;q)_\infty}$$

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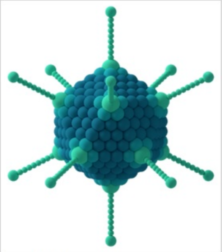
CAPSIDS



Icosahedral capsid of an Adenovirus

CAPSIDS

For a partition π let $\mu_k(\pi) = \mu_k$ be the number of parts that are equal to k.

Let $0 < k < m$. We say a partition π is an (m, k)-**capsid** if the only possible parts are $m - k$ or are congruent to 0 or k mod m, and satisfy the following condidtions:

- if $\mu_{m-k} = 0$, i.e., $m - k$ is not a part, then all parts are congruent to k mod m
- if $\mu_{m-k} > 0$, then $m - k$ is the smallest part and the largest part congruent to 0 mod m is $\leq m \cdot \mu_{m-k}$ and all parts congruent to k mod m (different from $m - k$, if $k = m/2$) are $> m \cdot \mu_{m-k}$.

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CAPSIDS

Let $\gamma(m,k,n)$ be the number of (m,k)-capsid partitions of n.
THEN

$$C_{m,k}(q) = 1 + \sum_{n=1}^{\infty} \gamma(m,k,n) q^n = \sum_{n=0}^{\infty} \frac{q^{(m-k)n}}{(q^m;q^m)_n (q^{mn+k};q^m)_\infty}.$$

Let $\mathcal{C}_{m,k}$ be the set of (m,k)-capsids.
EXAMPLE Let $m=5$ and $k=1$. There are seven $(5,1)$-capsid partitions of 16:

$$(1^{16}), \quad (4^4), \quad (1^{10},6), \quad (1^4,6^2), \quad (4,6^2), \quad (1^5,11), \quad (16) \in \mathcal{C}_{5,1},$$

and $a(5,1,16)=7$.
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CAPSIDS

q-BINOMIAL

$$\frac{(at;q)_\infty}{(a;q)_\infty(t;q)_\infty} = \sum_{n=0}^{\infty} \frac{t^n}{(q)_n(aq^n;q)_\infty}$$

$$C_{m,k}(q) = \frac{(q^m;q^m)_\infty}{(q^k;q^m)_\infty(q^{m-k};q^m)_\infty} = \frac{(q^m;q^m)_\infty}{P_{m,k}(q)},$$

where

$$P_{m,k}(q) := (q^k;q^m)_\infty(q^{m-k};q^m)_\infty.$$

ALSO

$$P_{m,k}(q) = P_{2m,k}(q)P_{2m,m-k}(q).$$

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$$\begin{aligned} C_{m,k}(q) &= \frac{(q^m;q^m)_\infty}{(q^k;q^m)_\infty(q^{m-k};q^m)_\infty} \\ &= \frac{(q^m;q^m)_\infty}{P_{m,k}(q)}, \end{aligned}$$

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$$P_{m,k}(q) := (q^k;q^m)_\infty(q^{m-k};q^m)_\infty.$$

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RAMANUJAN G-H-IDENTITY

$$1 = \frac{1}{P_{5,2}(q)P_{55,11}(q)} - \frac{q^2}{P_{5,1}(q)P_{55,22}(q)}$$

$$\begin{aligned}1 =& \frac{1}{P_{10,2}(q)P_{10,3}(q)P_{110,11}(q)P_{110,44}(q)} \\ &- \frac{q^2}{P_{10,1}(q)P_{10,4}(q)P_{110,22}(q)P_{110,33}(q)}\end{aligned}$$

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BACK TO TAU

Multiply by $q^{110}(q^{110};q^{110})_\infty^{24}$:

$$\begin{aligned} &q^{110}(q^{110};q^{110})_\infty^{24} \\ &= q^{110}\,\frac{(q^{10};q^{10})_\infty^2}{P_{10,2}(q)P_{10,3}(q)}\,\frac{(q^{110};q^{110})_\infty^2}{P_{10,1}(q^{11})P_{10,4}(q^{11})}\,\frac{(q^{110};q^{110})_\infty^{22}}{(q^{10};q^{10})_\infty^2} \\ &\quad - q^{112}\,\frac{(q^{10};q^{10})_\infty^2}{P_{10,1}(q)P_{10,4}(q)}\,\frac{(q^{110};q^{110})_\infty^2}{P_{10,2}(q^{11})P_{10,3}(q^{11})}\,\frac{(q^{110};q^{110})_\infty^{22}}{(q^{10};q^{10})_\infty^2} \end{aligned}$$

BACK TO TAU

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&\quad - q^{112}\,\frac{(q^{10};q^{10})_\infty^2}{P_{10,1}(q)P_{10,4}(q)}\,\frac{(q^{110};q^{110})_\infty^2}{P_{10,2}(q^{11})P_{10,3}(q^{11})}\,\frac{(q^{110};q^{110})_\infty^{22}}{(q^{10};q^{10})_\infty^2}
\end{aligned}
$$

BACK TO TAU

$$
\begin{aligned}
&\sum_{n=1}^{\infty} \tau(n) q^{110n} \\
&= q^{110}\, C_{10,2}(q)\, C_{10,3}(q)\, C_{10,1}(q^{11})\, C_{10,4}(q^{11})\, T_{11}(q^{10})^2 \\
&\quad - q^{112}\, C_{10,1}(q)\, C_{10,4}(q)\, C_{10,2}(q^{11})\, C_{10,3}(q^{11})\, T_{11}(q^{10})^2.
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BACK TO TAU

DEFINE

$$\mathcal{A} := \mathcal{C}_{10,2} \times \mathcal{C}_{10,3} \times \mathcal{C}_{10,1} \times \mathcal{C}_{10,4} \times \mathcal{T}_{11} \times \mathcal{T}_{11},$$
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For a partition π we let $|\pi|$ denote the sum of parts.
For $\vec{\pi} = (\pi_1, \pi_2, \pi_3, \pi_4, \pi_5, \pi_6)$ in $\mathcal{A}$ or in $\mathcal{B}$ DEFINE

$$|\vec{\pi}| := |\pi_1| + |\pi_2| + 11 \cdot |\pi_3| + 11 \cdot |\pi_4| + 10 \cdot |\pi_5| + 10 \cdot |\pi_6|.$$

If $|\vec{\pi}| = n$ we say that $\vec{\pi}$ is a vector partition of n. We let

$$a(n) := \text{the number of vector partitions in } \mathcal{A} \text{ of } n,$$

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BACK TO TAU

THEN

$$1+\sum_{n=1}^{\infty} a(n)q^n = C_{10,2}(q)\, C_{10,3}(q)\, C_{10,1}(q^{11})\, C_{10,4}(q^{11})\, T_{11}(q^{10})^2,$$

and

$$1+\sum_{n=1}^{\infty} b(n)q^n = C_{10,1}(q)\, C_{10,4}(q)\, C_{10,2}(q^{11})\, C_{10,3}(q^{11})\, T_{11}(q^{10})^2.$$

THEOREM [G & SCHLOSSER]

For $n \geq 1$, we have

$$(i) \qquad \tau(n) = a(110n-110) - b(110n-112),$$

and

$$(ii) \qquad 0 = a(n) - b(n-2), \qquad \text{if } n \not\equiv 0 \bmod 110.$$

BACK TO TAU

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$a_1(n) = a(110n - 100)$ $\qquad$ $b_1(n) = b(110n - 112)$

n	$a_1(n)$	$b_1(n)$	$\tau(n)$
1	1	0	1
2	174780	174804	-24
3	160427959	160427707	252
4	30666973078	30666974550	-1472
5	2476039898500	2476039893670	4830
6	113620092882699	113620092888747	-6048
7	3477782243224098	3477782243240842	-16744
8	78308735840958865	78308735840874385	84480
9	1383881468903570303	1383881468903683946	-113643
10	20083596309467339420	20083596309467455340	-115920

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6	113620092882699	113620092888747	-6048
7	3477782243224098	3477782243240842	-16744
8	78308735840958865	78308735840874385	84480
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