One Century of Quantum Mechanics

From de Broglie to Dirac (1924–1928)

Sergei K. Suslov

School of Mathematical & Statistical Sciences
Arizona State University

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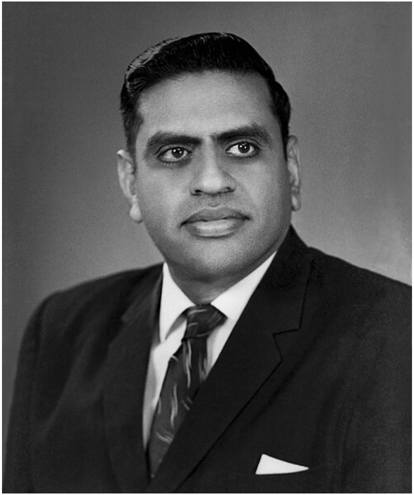
Alladi Ramakrishnan Memorial Lecture

THE 2026 GAINESVILLE INTERNATIONAL NUMBER THEORY CONFERENCE — **Alladian Festival** —
in honor of Krishna Alladi's 70th birthday

Dedicated to the memory of Professor Alladi Ramakrishnan

- Founder-Director of the **Institute of Mathematical Sciences** (IMSc), Chennai.
  `https://www.imsc.res.in`
- Pioneer in
  - stochastic processes
  - matrix algebra
  - particle physics
  - quantum theory
- His vision connected **mathematics and theoretical physics**.



1. **Origins of Quantum Theory**
   - Spectral lines of hydrogen
   - The first quantum revolution **(1911–1916)**: Rutherford, Bohr and Sommerfeld
2. **Wave Mechanics (1924–1926)**
   - de Broglie hypothesis
   - Schrödinger equation
   - Mathematical structure and exact solutions
3. **Relativistic Quantum Theory (1928)**
   - Dirac equation
   - Matrix methods and spin
4. **From Old Quantum Mechanics to Modern Physics**
   - Relativistic Kepler problems
   - Transuranic elements

The Hydrogen Spectrum: A Puzzle of the 19th Century

- In 1885 Johann Balmer discovered a remarkable numerical pattern in the spectral lines of hydrogen.
- The wavelengths satisfy the empirical relation

$$\frac{1}{\lambda} = R\left(\frac{1}{2^2} - \frac{1}{n^2}\right), \qquad n = 3, 4, 5, \ldots$$

First four lines (visible spectrum):

H_α (Red): $\lambda = 656\,\mathrm{nm}$ (transition $n = 3 \to n = 2$)

H_β (Blue–green): $\lambda = 486\,\mathrm{nm}$ (transition $n = 4 \to n = 2$)

H_γ (Blue): $\lambda = 434\,\mathrm{nm}$ (transition $n = 5 \to n = 2$)

H_δ (Violet): $\lambda = 410\,\mathrm{nm}$ (transition $n = 6 \to n = 2$).

BRIEF HISTORY:

- 1887 — **Albert A. Michelson** observes that the H_α spectral line is actually a doublet.
- 1897 — **Joseph J. Thomson** discovers the electron.
- 1911 — **Ernest Rutherford** proposes the nuclear model of the atom.
- 1913 — **Niels Bohr** introduces quantized electron orbits.
- 1916 — **Arnold Sommerfeld** extends Bohr's theory using relativistic elliptical orbits (the fine structure formula).
- 1926 — **Schrödinger equation** provides the nonrelativistic description of the hydrogen atom.
- 1928 — **Dirac equation** provides the correct relativistic description of the hydrogen atom.

This remarkable agreement with the fine structure formula became known as the *"Sommerfeld puzzle"*.

- French physicist and pioneer of quantum theory.
- Proposed the **wave nature of matter** (1923-24).
- Introduced the relation

$$\lambda = \frac{h}{p}$$

which connects particle momentum p with wavelength λ (h is Planck's constant).

- Nobel Prize in Physics (1929): https://www.nobelprize.org/prizes/physics/1929/summary/

Nobel Prize in Physics (1929):
"for the discovery of the wave nature of electrons."

Louis de Broglie (testimony)

"*After long reflection in solitude and meditation, I suddenly had the idea, during the year 1923, that the discovery made by Einstein in 1905 should be generalised by extending it to all material particles and notably to electrons.*"

– Louis de Broglie

The Einstein Connection: De Broglie generalized Einstein's 1905 idea that light behaves as particles (photons) by proposing the reverse: that particles (like electrons) behave as waves.

- One of the founders of **quantum mechanics**.
- Developed **matrix mechanics** (1925).
- Formulated the famous **uncertainty principle** (1927)

$$\Delta x \, \Delta p \geq \frac{\hbar}{2}$$

- Nobel Prize in Physics (1932): `https://www.nobelprize.org/prizes/physics/1932/summary/`

Nobel Prize in Physics (1932):
“for the creation of quantum mechanics.”

“*What we observe is not nature itself, but nature exposed to our method of questioning.*”

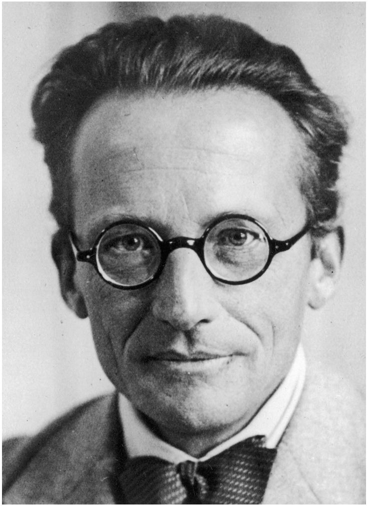
– Werner Heisenberg, Physics and Philosophy: The Revolution in Modern Science

ERWIN SCHRÖDINGER (1887–1961)

- Developed **wave mechanics** in 1926.
- Introduced the famous **Schrödinger equation**

$$i\hbar\frac{\partial\psi}{\partial t} = \widehat{H}\psi$$

- Provided a partial differential equation for quantum systems.
- Nobel Prize in Physics (1933): `https://www.nobelprize.org/prizes/physics/1933/summary/`



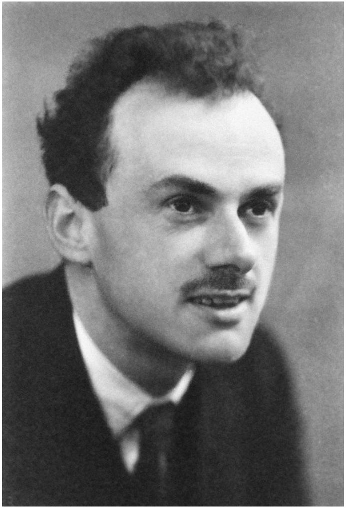
Nobel Prize in Physics (1933), shared with Paul Dirac.

Erwin Schrödinger (quote)

"*If a man never contradicts himself, the reason must be that he virtually never says anything at all.*"

— Erwin Schrödinger, What Is Life? with Mind and Matter and Autobiographical Sketches

- Formulated the **Dirac equation** (1928).
- Unified **quantum mechanics and special relativity**.
- Predicted the existence of **antimatter**.
- One of the founders of **quantum field theory**.
- Nobel Prize in Physics (1933): `https://www.nobelprize.org/prizes/physics/1933/summary/`



Nobel Prize in Physics (1933), shared with Erwin Schrödinger.

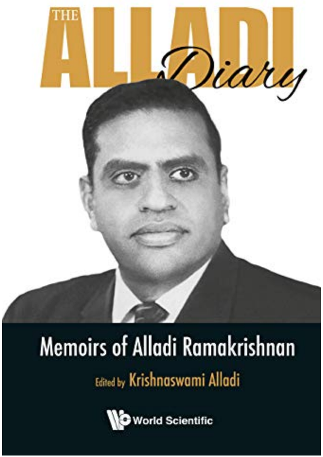
"A physical law must possess mathematical beauty."

– Paul Dirac

During a seminar at Moscow State University on October 3, 1956, when asked to summarize his philosophy of physics, Dirac wrote the above-cited sentence on the blackboard in capital letters.

- Niels Bohr
- Werner Heisenberg
- Paul Dirac
- Hideki Yukawa
- Abdus Salam
- Hans Bethe
- Chen Ning Yang
- Robert Oppenheimer
- Richard Feynman
- Murray Gell-Mann
- Subrahmanyan Chandrasekhar

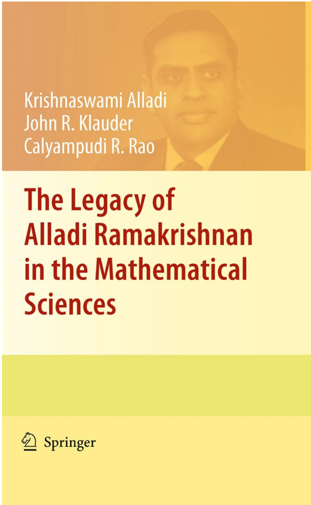
These scientists interacted with **Alladi Ramakrishnan** including his *Theoretical Physics Seminar* in Madras.



Alladi Diary: Memoirs of Alladi Ramakrishnan, World Scientific (2019)

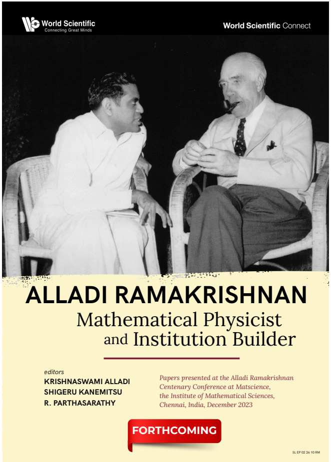
edited by Krishna Alladi

To the Memory of Professor Alladi Ramakrishnan



The Legacy of Alladi Ramakrishnan in the Mathematical Sciences, Springer (2010)

edited by K. Alladi et al



Alladi Ramakrishnan: Mathematical Physicist and Institution Builder, World Scientific

edited by K. Alladi et al

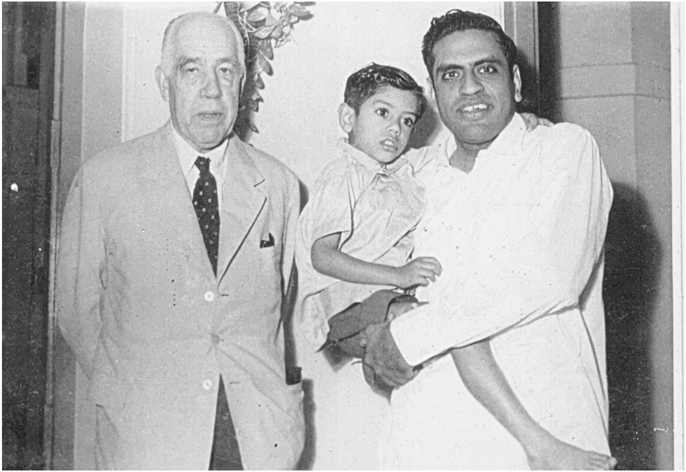
Niels Bohr and Alladi Ramakrishnan

Nobel Laureate Niels Bohr in discussion with Professor Alladi Ramakrishnan at Ekamra Nivas, January 1960.

Niels Bohr and Mrs. Bohr with the Ramakrishnan Family

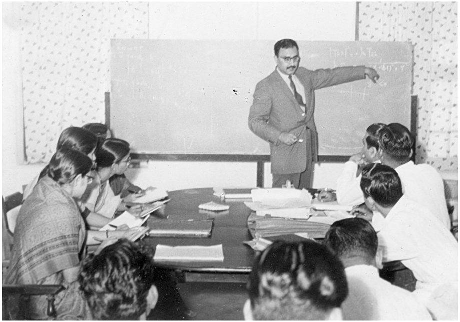
Niels Bohr and Mrs. Bohr with Alladi and Lalitha Ramakrishnan at Ekamra Nivas, January 1960.

Niels Bohr, Alladi Ramakrishnan and Krishna Alladi

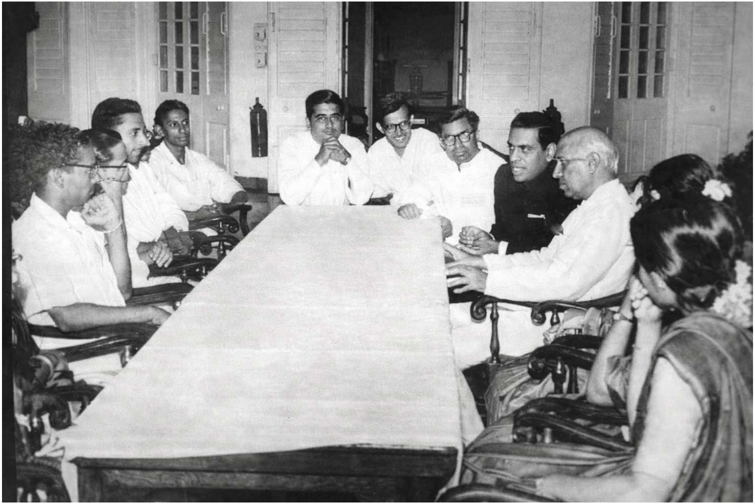


Krishna Alladi with his father Alladi Ramakrishnan and Nobel Laureate Niels Bohr at Ekamra Nivas, January 1960.

- Nobel Laureate **Abdus Salam** participated in Alladi Ramakrishnan's theoretical physics seminar.
- These seminars attracted many leading physicists of the twentieth century.
- They played an important role in the development of theoretical physics in India.



Professor Abdus Salam at the Theoretical Physics Seminar, Ekamra Nivas, January 1960.



Professor Alladi Ramakrishnan and the students of his Theoretical Physics Seminar with Prime Minister Jawaharlal Nehru at Raj Bhavan, Madras, October 8, 1961.

Indira Gandhi, Alladi Ramakrishnan and Krishna Alladi

Alladi Ramakrishnan, Prime Minister Indira Gandhi and Madras Chief Minister Bhakthavatsalam at MATSCIENCE, January 1967.

Eleven year old Krishna presenting a bouquet to Prime Minister Indira Gandhi at MATSCINCE.

George Andrews at the Alladi Foundation

Professor George Andrews lecturing at the Alladi Foundation during Ramanujan's Centenary, December 23, 1987.

Alladi Ramakrishnan with Outstanding Mathematicians

International visitors at Ekamra Nivas during Ramanujan's Centenary celebration, December 1987.

1924-26 Revolution: Schrödinger Equation – "A Joke of Nature"

Freeman Dyson, *Birds and Frogs*, Notices of the AMS **56** (2009), 212–223.

`https://pdodds.w3.uvm.edu/files/papers/others/2009/dyson2009a.pdf`

"One of the most profound jokes of nature is the square root of minus one that Schrödinger put into his wave equation when he invented wave mechanics in 1926."

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = \widehat{H}\psi$$

Hamiltonian, energy operator:

$$\widehat{H} = \frac{\widehat{\mathbf{p}}^2}{2m} + U(\mathbf{r}), \qquad \widehat{\mathbf{p}} = \frac{\hbar}{\mathrm{i}}\nabla = \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial \mathbf{r}}$$

Time-dependent Schrödinger equation, a new class of PDEs:

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + U\psi$$

"And suddenly it became a new kind of wave equations instead of a heat conduction equation."

If the potential is time independent $U = U(\mathbf{r})$ we separate variables

$$\psi(\mathbf{r}, t) = e^{-\mathrm{i}Et/\hbar}\chi(\mathbf{r})$$

which leads to the **stationary Schrödinger equation**:

$$-\frac{\hbar^2}{2m}\Delta\chi + U\chi = E\chi$$

- **Eigenvalue problem** for the Hamiltonian:

$$\widehat{H}\chi = E\chi$$

- Born's **probabilistic interpretation**:

$$|\psi|^2$$

probability density with normalization

$$\int_{\mathbb{R}^3} |\psi|^2 \, dv = 1$$

"This equation describes correctly everything we know about the behavior of atoms. It is the basis of chemistry and most of quantum physics." (Freeman Dyson)

Two approaches:

- **Numerical solutions and mathematical models**
- **Exact solutions using special functions**

Flügge S., *Practical Quantum Mechanics*, Springer (1999)

https://archive.org/details/PracticalQuantumMechanicsS.Flgge/page/n1/mode/2up

Computer algebra support:
Ellis L., Ellis I., Koutschan C., Suslov S. K.
On Potentials Integrated by the Nikiforov–Uvarov Method

https://bookstore.ams.org/view?ProductCode=CONM/819

A Remarkable Place in the History of Physics

Villa Frisia of Dr. Herwig's sanatorium, Arosa (Switzerland), where Schrödinger stayed during the winter of 1925–1926 while developing wave mechanics.

Postcard view showing the Villa Frisia (right), where Schrödinger stayed, and the house of Dr. Otto Herwig (left).

The "**Mecca of Quantum Physics**":

The author visiting Villa Frisia in Arosa on December 29, 2025, the historic location associated with the discovery of Schrödinger's wave equation.

J. Mehra and H. Rechenberg, *The Historical Development of Quantum Theory*, Vol. 5: *The Creation of Wave Mechanics: Early Response and Applications (1925–1926)*, Springer, New York, 1987. https://www.amazon.com/Mechanics-Applications-1925-1926-Historical-Development/dp/0387963774

K. Barley, J. Vega-Guzman, A. Ruffing, and S. K. Suslov, *Discovery of the relativistic Schrödinger equation*, Physics–Uspekhi **65** (2022), 90–103. https://iopscience.iop.org/article/10.3367/UFNe.2021.06.039000/pdf

L. Ellis, I. Ellis, C. Koutschan, and S. K. Suslov, *On Potentials Integrated by the Nikiforov–Uvarov Method*, in: Applications and *q*-Extensions of Hypergeometric Functions, H. S. Cohl, R. S. Costas-Santos, and R. S. Maier, Editors, Contemporary Mathematics **819** (2025), 43–95. https://bookstore.ams.org/view?ProductCode=CONM/819

K. Barley, A. Ruffing, and S. K. Suslov, *Old quantum mechanics by Bohr and Sommerfeld from a modern perspective*, Physics–Uspekhi **69** (2026), 74–93. https://ufn.ru/en/articles/2026/1/e/

Heisenberg uncertainty relation:

$$\Delta x \, \Delta p \geq \frac{\hbar}{2}$$

- **Remark 1.** Schrödinger coherent states — Heisenberg limit:

$$\Delta x = \Delta p = \text{constant}$$

Coherent States

- **Remark 2.** Minimum-uncertainty squeezed states — how to go below the symmetric Heisenberg limit ("missing" solutions).

Kryuchkov S. I., Suslov S. K., Vega-Guzman J. M., *The minimum-uncertainty squeezed states for atoms and photons in a cavity*, J. Phys. B **46** (2013) 104007.

(IOP SELECT and HIGHLIGHT for 2013) `https://iopscience.iop.org/article/10.1088/0953-4075/46/10/104007/pdf`

The time-dependent Schrödinger equation for the simple harmonic oscillator in one dimension (dimensionless units):

$$2\mathrm{i}\psi_t + \psi_{xx} - x^2\psi = 0$$

is invariant under the following transformation:

$$\psi(x,t) = \sqrt{\beta(t)}\ e^{\mathrm{i}S(x,t)}\chi(\xi,\tau),$$
$$S(x,t) = \alpha(t)x^2 + \delta(t)x + \kappa(t)$$
$$\xi = \beta(t)x + \varepsilon(t),\ \tau = -\gamma(t)$$

with respect to new variables χ and ξ, τ:

$$2\mathrm{i}\chi_\tau + \chi_{\xi\xi} - \xi^2\chi = 0$$

for certain time-dependent functions $\alpha, \beta, \gamma, \delta, \varepsilon, \kappa$.

Here

$$\alpha(t) = \frac{\alpha_0 \cos 2t + \sin 2t(\beta_0^4 + 4\alpha_0^2 - 1)/4}{\beta_0^4 \sin^2 t + (2\alpha_0 \sin t + \cos t)^2}$$

$$\beta(t) = \frac{\beta_0}{\sqrt{\beta_0^4 \sin^2 t + (2\alpha_0 \sin t + \cos t)^2}}$$

$$\gamma(t) = \gamma_0 - \frac{1}{2} \arctan \frac{\beta_0^2 \tan t}{1 + 2\alpha_0 \tan t}$$

$$\delta(t) = \frac{\delta_0(2\alpha_0 \sin t + \cos t) + \varepsilon_0 \beta_0^3 \sin t}{\beta_0^4 \sin^2 t + (2\alpha_0 \sin t + \cos t)^2}$$

$$\varepsilon(t) = \frac{\varepsilon_0(2\alpha_0 \sin t + \cos t) - \beta_0 \delta_0 \sin t}{\sqrt{\beta_0^4 \sin^2 t + (2\alpha_0 \sin t + \cos t)^2}}$$

and

$$\kappa(t) = \kappa_0 + \sin^2 t \frac{\varepsilon_0 \beta_0^2 (\alpha_0 \varepsilon_0 - \beta_0 \delta_0) - \alpha_0 \delta_0^2}{\beta_0^4 \sin^2 t + (2\alpha_0 \sin t + \cos t)^2} + \frac{1}{4} \sin 2t \frac{\varepsilon_0^2 \beta_0^2 - \delta_0^2}{\beta_0^4 \sin^2 t + (2\alpha_0 \sin t + \cos t)^2}$$

where $\alpha_0, \beta_0 \neq 0, \gamma_0, \delta_0, \varepsilon_0, \kappa_0$ are real-valued constants.

"Missing" solutions:

$$\psi_n(x,t) = e^{\mathrm{i}(\alpha x^2 + \delta x + \kappa) + \mathrm{i}(2n+1)\gamma} \sqrt{\frac{\beta}{2^n n! \sqrt{\pi}}} \times e^{-\xi^2/2} H_n(\xi), \qquad \xi = \beta x + \varepsilon$$

by action of the Schrödinger group on standard textbook solutions.[1]

[1] $\beta_0 = 1, \alpha_0 = \gamma_0 = \delta_0 = \varepsilon_0 = \kappa_0 = 0$ for textbook solutions.

Ehrenfest theorem for the "missing" Gaussian wave packet ($n = 0$):

$$\frac{d}{dt}\langle\widehat{x}\rangle = \langle\widehat{p}\rangle, \qquad \frac{d}{dt}\langle\widehat{p}\rangle = -\langle\widehat{x}\rangle$$

Expectation values (harmonic motion):

$$\langle\widehat{x}\rangle = -\frac{1}{\beta_0}\left[(2\alpha_0\varepsilon_0 - \beta_0\delta_0)\sin t + \varepsilon_0\cos t\right]$$

$$\langle\widehat{p}\rangle = -\frac{1}{\beta_0}\left[(2\alpha_0\varepsilon_0 - \beta_0\delta_0)\cos t + \varepsilon_0\sin t\right]$$

For the variances:

$$\langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle = \frac{1}{16\beta_0^4}\Big[(\beta_0^4 + 4\alpha_0^2 + 1)^2 - ((\beta_0^4 + 4\alpha_0^2 - 1)\cos 2t - 4\alpha_0\sin 2t)^2\Big]$$

Upper and lower bounds in the uncertainty relation:

$$\max\langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle = \frac{(\beta_0^4 + 4\alpha_0^2 + 1)^2}{16\beta_0^4}$$

$$\min\langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle = \frac{1}{4}$$

These extrema occur when one variance is minimal and the other maximal, which explicitly demonstrates **squeezing.**

In the creator's own words – Heisenberg (1930):

"If the classical motion of the system is periodic, the size of the wave packet may undergo periodic changes."

Mathematica animation: *HarmonicOscillatorGroup.nb*

Relativistic wave equation for a spin-$1/2$ particle:

$$\mathrm{i}\hbar \frac{\partial}{\partial t}\psi(\mathbf{r}, t) = \widehat{H}\psi(\mathbf{r}, t)$$

where ψ is a complex four-component wave function.
Standard Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Dirac matrices:

$$\boldsymbol{\alpha} = \begin{pmatrix} O & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & O \end{pmatrix}, \qquad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

Hamiltonian operator:

$$\widehat{H} = c(\boldsymbol{\alpha} \cdot \widehat{\boldsymbol{p}}) + mc^2\beta$$

with

$$\widehat{\boldsymbol{p}} = -\mathrm{i}\hbar\nabla$$

Four-component wave function (bi-spinor):

$$\psi(\mathbf{r}, t) = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

The Dirac equation represents a system of four first-order PDEs.
Plane wave solution:

$$\psi(\mathbf{r}, t) = e^{\frac{\mathrm{i}}{\hbar}(\mathbf{p}\cdot\mathbf{r} - Et)} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$$

where u is a constant bi-spinor.

Eigenvalue conditions – integrals of motion:

$$\mathrm{i}\hbar \frac{\partial}{\partial t}\psi = E\psi$$

$$\widehat{\boldsymbol{p}}\psi = -\mathrm{i}\hbar\nabla\psi = \boldsymbol{p}\,\psi$$

Substitution into the Dirac equation gives

$$(c\,\boldsymbol{\alpha}\cdot\mathbf{p} + mc^2\beta)u = Eu$$

Block matrix form:

$$\begin{pmatrix} (mc^2 - E)I & c\,\boldsymbol{\sigma}\cdot\mathbf{p} \\ c\,\boldsymbol{\sigma}\cdot\mathbf{p} & -(mc^2 + E)I \end{pmatrix} u = 0$$

This representation simplifies computation of eigenvalues and eigenvectors using matrix algebra.

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Formulas/equations are smarter than people:

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Formulas/equations are smarter than people:
One cannot escape the feeling that these mathematical formulae have an independent existence and an intelligence of their own, that they are wiser than we are, wiser even than their discoverers, that we get more out of them than was originally put into them.

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(Heinrich Hertz, on Maxwell's equations of electromagnetism)

We combine the eigenvectors into a *single matrix form*, an idea introduced by **Alladi Ramakrishnan**:

A. Ramakrishnan, *The Dirac Hamiltonian as a Member of Hierarchy of Matrices*, J. Math. Anal. Appl. **20** (1967), 9–16.

A. Ramakrishnan, *L-Matrix Theory: The Grammar of Dirac Matrices*, Tata McGraw–Hill Publishing Company, 1972.

The eigenvalue problem can be solved using the identity ($E \to -E$):

$$\begin{pmatrix} (mc^2 - E)I & c\,\boldsymbol{\sigma}\cdot\mathbf{p} \\ c\,\boldsymbol{\sigma}\cdot\mathbf{p} & -(mc^2 + E)I \end{pmatrix} \begin{pmatrix} (mc^2 + E)I & c\,\boldsymbol{\sigma}\cdot\mathbf{p} \\ c\,\boldsymbol{\sigma}\cdot\mathbf{p} & -(mc^2 - E)I \end{pmatrix}$$

$$= (m^2c^4 + c^2\mathbf{p}^2 - E^2) \begin{pmatrix} I & O \\ O & I \end{pmatrix}$$

Thus the free particle solutions lie on the spectrum of the Dirac Hamiltonian: $E = \pm\sqrt{m^2c^4 + c^2\mathbf{p}^2}$.

This matrix approach avoids evaluation of the full 4×4 determinant in the eigenvalue problem.

Consider the “mixed” partitioned matrix:

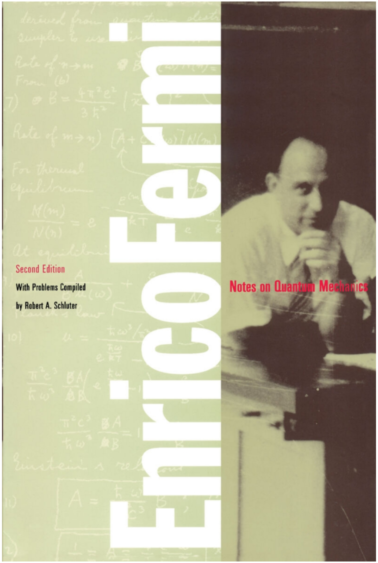
$$\begin{pmatrix} (mc^2+E)I & c\,\boldsymbol{\sigma}\cdot\mathbf{p} \\ c\,\boldsymbol{\sigma}\cdot\mathbf{p} & -(mc^2+E)I \end{pmatrix}^2 = \left((mc^2+E)^2 + c^2\mathbf{p}^2\right)\begin{pmatrix} I & O \\ O & I \end{pmatrix}$$

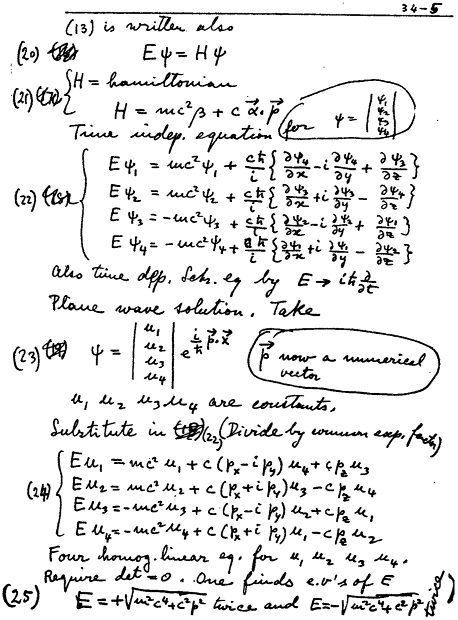
These identities yield the normalized eigenvectors from the book:

- A. Ramakrishnan, *Elementary Particles and Cosmic Rays*, Pergamon Press, New York, 1962. https://archive.org/details/elementarypartic00ramauoft/page/ii/mode/2up
- E. Fermi, *Notes on Quantum Mechanics*, Second Edition, University of Chicago Press, Chicago and London, 1961(First Edition), 1995(Second Edition).
- S. K. Suslov and K. Alladi, *How Enrico Fermi Could Have Avoided an Oversight in His Last Lecture Notes on the Dirac Theory of Free Electron*, World Scientific, submitted.

Handwritten lecture notes prepared by **Enrico Fermi** for his final course at the University of Chicago in 1954.

They demonstrate his remarkable clarity in presenting the foundations of quantum mechanics.





34-5

(13) is written also

(20) $E\psi = H\psi$

(21) H = hamiltonian

$H = mc^2\beta + c\,\vec{\alpha}\cdot\vec{p}$ (for $\psi = \begin{vmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{vmatrix}$)

Time indep. equation

(22)
$$
\begin{cases}
E\psi_1 = mc^2\psi_1 + \frac{c\hbar}{i}\left\{\frac{\partial\psi_4}{\partial x} - i\frac{\partial\psi_4}{\partial y} + \frac{\partial\psi_3}{\partial z}\right\}\\
E\psi_2 = mc^2\psi_2 + \frac{c\hbar}{i}\left\{\frac{\partial\psi_3}{\partial x} + i\frac{\partial\psi_3}{\partial y} - \frac{\partial\psi_4}{\partial z}\right\}\\
E\psi_3 = -mc^2\psi_3 + \frac{c\hbar}{i}\left\{\frac{\partial\psi_2}{\partial x} - i\frac{\partial\psi_2}{\partial y} + \frac{\partial\psi_1}{\partial z}\right\}\\
E\psi_4 = -mc^2\psi_4 + \frac{c\hbar}{i}\left\{\frac{\partial\psi_1}{\partial x} + i\frac{\partial\psi_1}{\partial y} - \frac{\partial\psi_2}{\partial z}\right\}
\end{cases}
$$

Also time dep. Schr. eq by $E \to i\hbar\frac{\partial}{\partial t}$

Plane wave solution. Take

(23) $\psi = \begin{vmatrix}u_1\\ u_2\\ u_3\\ u_4\end{vmatrix} e^{\frac{i}{\hbar}\vec{p}\cdot\vec{x}}$ ($\vec{p}$ now a numerical vector)

u_1, u_2, u_3, u_4 are constants.

Substitute in (22) (Divide by common exp. factor)

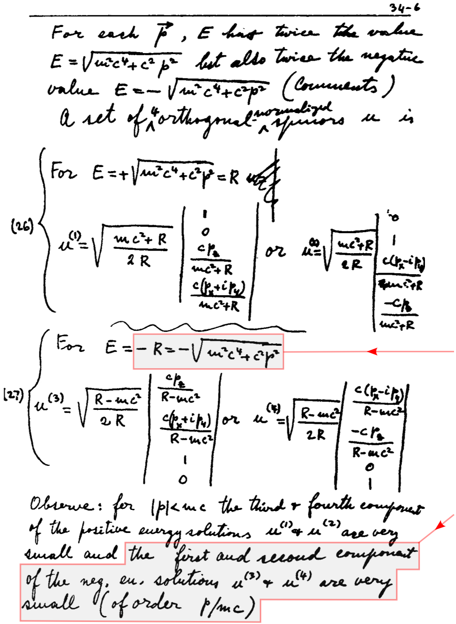
(24)
$$
\begin{cases}
Eu_1 = mc^2u_1 + c(p_x - ip_y)u_4 + cp_zu_3\\
Eu_2 = mc^2u_2 + c(p_x + ip_y)u_3 - cp_zu_4\\
Eu_3 = -mc^2u_3 + c(p_x - ip_y)u_2 + cp_zu_1\\
Eu_4 = -mc^2u_4 + c(p_x + ip_y)u_1 - cp_zu_2
\end{cases}
$$

Four homog. linear eq. for u_1, u_2, u_3, u_4.

Require det = 0. One finds e.v.'s of E

(25) $E = +\sqrt{m^2c^4 + c^2p^2}$ twice and $E = -\sqrt{m^2c^4 + c^2p^2}$ twice)

Page 34–5 of Fermi's lecture notes discussing Dirac's equation and the associated eigenvalue problem.



34-6

For each $\vec{p}$, E has twice the value $E=\sqrt{m^2c^4+c^2p^2}$ but also twice the negative value $E=-\sqrt{m^2c^4+c^2p^2}$ (Comments)

A set of 4 orthogonal normalized spinors u is

(26) For $E=+\sqrt{m^2c^4+c^2p^2}=R$

$$u^{(1)}=\sqrt{\frac{mc^2+R}{2R}}\begin{vmatrix}1\\0\\\frac{cp_z}{mc^2+R}\\\frac{c(p_x+ip_y)}{mc^2+R}\end{vmatrix}\quad\text{or}\quad u^{(2)}=\sqrt{\frac{mc^2+R}{2R}}\begin{vmatrix}0\\1\\\frac{c(p_x-ip_y)}{mc^2+R}\\\frac{-cp_z}{mc^2+R}\end{vmatrix}$$

(27) For $E=-R=-\sqrt{m^2c^4+c^2p^2}$

$$u^{(3)}=\sqrt{\frac{R-mc^2}{2R}}\begin{vmatrix}\frac{cp_z}{R-mc^2}\\\frac{c(p_x+ip_y)}{R-mc^2}\\1\\0\end{vmatrix}\quad\text{or}\quad u^{(4)}=\sqrt{\frac{R-mc^2}{2R}}\begin{vmatrix}\frac{c(p_x-ip_y)}{R-mc^2}\\\frac{-cp_z}{R-mc^2}\\0\\1\end{vmatrix}$$

Observe: for $|p|\ll mc$ the third & fourth component of the positive energy solutions $u^{(1)}$ & $u^{(2)}$ are very small and the first and second component of the neg. en. solutions $u^{(3)}$ & $u^{(4)}$ are very small (of order p/mc)

Page 34–6 of Fermi's notes: both bi-spinors (27) correspond to

$$E=+R$$

Although Fermi's bi-spinors are normalized, they are not mutually orthogonal:

$$\left(u^{(1)}\right)^{\dagger} u^{(3)} = \frac{p_3}{|\mathbf{p}|}, \qquad \left(u^{(2)}\right)^{\dagger} u^{(3)} = \frac{p_1 + i p_2}{|\mathbf{p}|}$$

Hence

$$u^{(3)} = \frac{p_3}{|\mathbf{p}|} u^{(1)} + \frac{p_1 + i p_2}{|\mathbf{p}|} u^{(2)}$$

$$u^{(4)} = \frac{p_1 - i p_2}{|\mathbf{p}|} u^{(1)} - \frac{p_3}{|\mathbf{p}|} u^{(2)}$$

Fermi also noted:

"for $|p| \ll mc$, the third and fourth components of the positive energy solutions $u^{(1)}$ and $u^{(2)}$ are very small; and the first and second components of the negative energy solutions $u^{(3)}$ and $u^{(4)}$ are also very small (on the order of p/mc)".

This holds only for the first two bi-spinors (as already noticed, all four eigenvectors belong to the positive eigenvalue).

In the limit $c \to \infty$:

$$(u^{(3)}, u^{(4)}) \to \begin{pmatrix} \dfrac{\boldsymbol{\sigma} \cdot \mathbf{p}}{|\mathbf{p}|} \\ O \end{pmatrix}$$

Gaussian elimination gives

$$\begin{pmatrix} (mc^2 + E)I & c\,\sigma \cdot \mathbf{p} \\ c\,\sigma \cdot \mathbf{p} & -(mc^2 - E)I \end{pmatrix} \to \begin{pmatrix} (mc^2 + E)I & c\,\sigma \cdot \mathbf{p} \\ O & O \end{pmatrix}$$

Hence the matrix rank is two.

Only two linearly independent rows and columns remain, revealing the degeneracy structure of the Dirac solutions.

Following **Eugene Wigner**, the Poincaré group provides the natural framework for describing the mass and spin of elementary particles. The Pauli–Lubański pseudovector is defined by

$$w_\mu = \frac{1}{2}\,\varepsilon_{\mu\nu\sigma\tau}\, p^\nu M^{\sigma\tau}, \qquad p_\mu w^\mu = 0$$

Here

- p_μ is the relativistic four-momentum operator,
- $M^{\sigma\tau}$ are the generators of the Lorentz group.

The mass and spin follow from the two Casimir invariants of the Poincaré group:

$$p^2 = p_\mu p^\mu = m^2, \qquad w^2 = w_\mu w^\mu = -m^2 s(s+1)$$

We work in Minkowski space-time $\mathbb{R}^4$ using natural units $c = \hbar = 1$.

Consider a free relativistic particle with mass m and spin $1/2$ described by a bispinor wave function ψ.
Our analysis shows that

$$w_\mu = \frac{1}{2}\left(p_\mu + m\gamma_\mu\right)\gamma_5$$

provided the Dirac equation holds

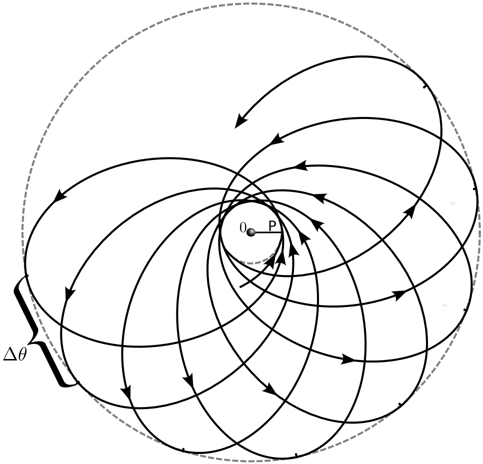
$$(\gamma^\mu p_\mu - m)\psi = 0$$

This relation immediately implies

$$s = \frac{1}{2}$$

in covariant form.

S. I. Kryuchkov, N. A. Lanfear, and S. K. Suslov, *The role of the Pauli–Lubanski vector for the Dirac, Weyl, Proca, Maxwell, and Fierz–Pauli equations*, Physica Scripta **91** (2016), 035301.
https://iopscience.iop.org/article/10.1088/0031-8949/91/3/035301



Relativistic Kepler motion. The nucleus is located at the fixed focus O. The perihelion P evolves while the perihelion and aphelion move along two concentric circles around O.

Arnold Sommerfeld and Niels Bohr

Arnold Sommerfeld and Niels Bohr in Lund, September 1919, during Sommerfeld's lecture tour of Scandinavia after World War I. Courtesy of Deutsches Museum, Munich.

Orbital equation in geometrical form:

$$\frac{1}{r} = \frac{1 + \epsilon \cos(\omega\theta)}{a(1 - \epsilon^2)}$$

Perihelion and aphelion distances:

$$r_{\min} = a(1 - \epsilon), \qquad r_{\max} = a(1 + \epsilon)$$

Sommerfeld's fine-structure formula for hydrogen-like atoms:

$$\frac{E_{n_r, n_\theta}}{mc^2} = \left[1 + \frac{\alpha^2 Z^2}{\left(n_r + \sqrt{n_\theta^2 - \alpha^2 Z^2}\right)^2}\right]^{-1/2} \qquad (Z < 137)$$

Here

- n_r — radial quantum number ($n_r = 0, 1, 2, 3, \ldots$)
- n_θ — azimuthal quantum number ($n_\theta = 1, 2, 3, \ldots$)
- $\alpha = e^2/(\hbar c) \approx 1/137$

This formula explained the fine structure of spectral lines.

Quantized parameters of the electron's elliptical orbits:

$$\omega_{n_\theta} n_\theta = \sqrt{n_\theta^2 - \alpha^2 Z^2}$$

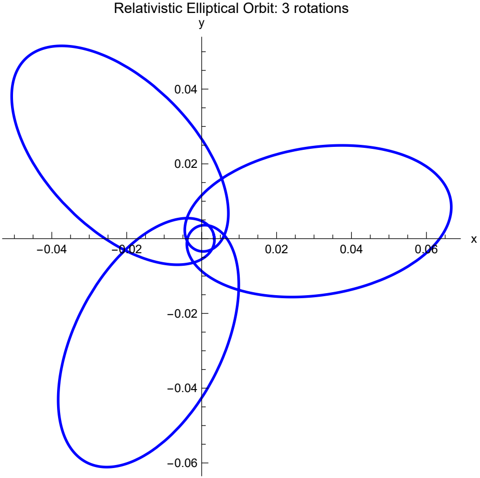
$$\epsilon_{n_r,n_\theta} = \sqrt{n_r}\,\frac{\sqrt{n_r + 2\sqrt{n_\theta^2 - \alpha^2 Z^2}}}{n_r + \sqrt{n_\theta^2 - \alpha^2 Z^2}}$$

$$a_{n_r,n_\theta} = \frac{a_0}{Z}\left(n_r + \sqrt{n_\theta^2 - \alpha^2 Z^2}\right) \times \sqrt{\alpha^2 Z^2 + \left(n_r + \sqrt{n_\theta^2 - \alpha^2 Z^2}\right)^2}$$

where $a_0 = \frac{\hbar^2}{me^2}$ is the Bohr radius and $\alpha = \frac{e^2}{\hbar c} \approx \frac{1}{137}$ is the fine-structure constant.

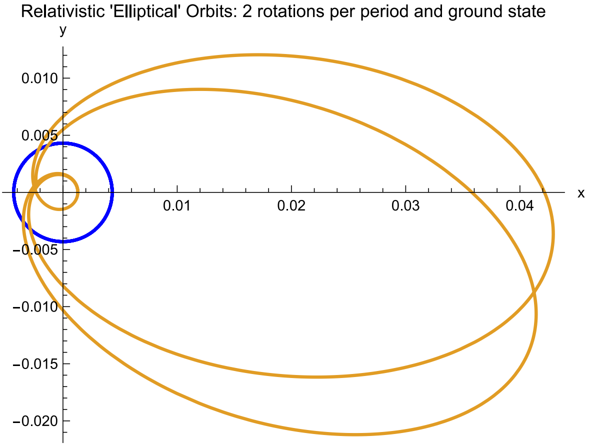
First Quantum Revolution (1913–1916).

Uranium ($Z = 92$)



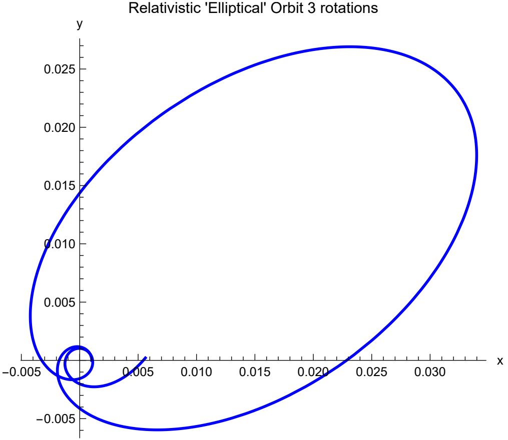
Kepler elliptical motion in the hydrogen-like ion U^{91+}.

Oganesson ($Z = 118$)



Clockwise Kepler motion with self-intersection in the hydrogen-like ion Og^{117+}.

Unbibium ($Z = 122$)



Kepler-type orbit with four self-intersections in the hypothetical ion Ubb^{121+}.

- In the framework of "old quantum mechanics", we observe that in ultra-strong Coulomb fields (such as for Oganesson) electron trajectories may become *self-intersecting*.[2]
- Following Mendeléev's idea of obtaining "much from little", one may also investigate rotational scattering of electrons in super-heavy Coulomb fields and a similar relativistic effects in astrophysics.
- This may also lead to interesting consequences in quantum electrodynamics of super-heavy transuranics and possibly in super-strong gravitational fields in general relativity.

Examples (computer algebra animations):

Coherent states vs squeezed states: *HarmonicOscillatorGroup.nb*

Semi-classical Kepler motion: *EllipsesAnimateAu.nb*

Transuranics: *Uranium vs Oganesson and beyond*

[2] J. A. Wheeler referred to such trajectories as "chemical orbits" or "double necklaces".

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Keep moving forward!

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Happy 70th birthday to Krishna!

Keep moving forward!

Carrying on father's legacy!

Happy Birthday! (Oct 5, 2025)

With best wishes, deepest respect and gratitude for all your numerous achievements! (Friday, Sept 4, 2026)