

A simple proof of the Atkin-O'Brien partition congruence conjecture for powers of 13

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Abstract

In 1967, Atkin and O'Brien conjectured congruences for the partition function involving Hecke operators modulo powers of 13. In this paper, we provide a simple proof of this conjecture.

1 Introduction

The partition function $p(n)$ counts the number of distinct ways of expressing a positive integer n as a sum of positive integers, without considering the order of the summands. For example, the partitions of 4 are:

$$4, \quad 3 + 1, \quad 2 + 2, \quad 2 + 1 + 1, \quad 1 + 1 + 1 + 1,$$

so $p(4) = 5$.

The generating function for the partition function is an infinite product given by:

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{k=1}^{\infty} \frac{1}{1 - q^k}, \quad (1)$$

where $|q| < 1$.

The q -Pochhammer symbol is defined as:

$$(a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k), \quad (2)$$

and for $n = \infty$:

$$(a; q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k). \quad (3)$$

Using this notation, the generating function for the partition function can be concisely written as:

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q; q)_\infty} = \frac{1}{f(q)}. \quad (4)$$

The Dedekind eta function is a modular form defined in the upper half-plane \mathbb{H} by:

$$\eta(\tau) = e^{\pi i \tau / 12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau}), \quad (5)$$

where $\tau \in \mathbb{H}$. Setting $q = e^{2\pi i \tau}$, we have $|q| < 1$, so the product converges. Using the q -Pochhammer symbol, the eta function can be expressed as:

$$\eta(\tau) = q^{1/24} (q; q)_\infty, \quad (6)$$

where $q^{1/24} = e^{\pi i \tau / 12}$. Therefore, the reciprocal of the eta function is:

$$\eta(\tau)^{-1} = q^{-1/24} \frac{1}{(q; q)_\infty}, \quad (7)$$

which is a weakly holomorphic modular form on $M_{-1/2}^!(1, \nu_\eta^{-1})$, where ν_η is the eta multiplier. From equation (4), we see that:

$$\frac{1}{(q; q)_\infty} = \sum_{n=0}^{\infty} p(n)q^n. \quad (8)$$

Thus, the generating function of $p(n)$ is related to the Dedekind eta function by:

$$\eta(\tau)^{-1} = q^{-1/24} \sum_{n=0}^{\infty} p(n)q^n = \sum_{n=-1}^{\infty} p\left(\frac{n+1}{24}\right) q^{n/24}, \quad (9)$$

where $p(n) = 0$ if n is not a positive integer.

In this paper we will prove a conjecture stated by Atkin and O'Brien in [3]. We will follow their work and let

$$P(N) = \begin{cases} p(n) & \text{if } N = 24n - 1, \\ 0 & \text{if } N < -1 \text{ or } N \not\equiv -1 \pmod{24} \text{ or } N \text{ is nonintegral.} \end{cases}$$

In their work, they proved

Theorem 1.1. *For all $\alpha \geq 1$ there exists an integral constant K_α not divisible by 13 such that for all N*

$$P(13^{\alpha+2}N) \equiv K_\alpha P(13^\alpha N) \pmod{13^\alpha}. \quad (10)$$

They also made the following conjecture

Conjecture 1.2. *Let $\alpha \geq 1$, and $p \neq 13$ be a prime ≥ 5 . Then there exists a constant $k = k(p, \alpha)$ such that for all N ,*

$$P(p^2 \cdot 13^\alpha N) - \left\{ k - \left(\frac{-3 \cdot 13^\alpha N}{p} \right) p^{-2} \right\} P(13^\alpha N) + p^{-3} P\left(\frac{13^\alpha N}{p^2}\right) \equiv 0 \pmod{13^\alpha},$$

where $\left(\frac{a}{b}\right)$ is the Jacobi symbol.

Note: Atkin and O'Brien proved the conjecture for $\alpha = 1, 2$.

An immediate corollary is

Corollary 1.3. *For $\alpha \geq 1$ and for primes $p \geq 5$, $p \neq 13$, there is a constant $k = k(p, \alpha)$ such that for all n coprime to p*

$$p \left(\frac{13^\alpha p^3 n + 1}{24} \right) \equiv k p \left(\frac{13^\alpha p n + 1}{24} \right) \pmod{13^\alpha}.$$

In [5], the authors proved results similar to Theorem 1.1, Conjecture 1.2, and Corollary 1.3 for powers of all primes ℓ such that $5 \leq \ell \leq 31$. Namely,

Theorem 1.4. *Suppose that $5 \leq \ell \leq 31$ is a prime, and that $\alpha \geq 1$. If $b_1 \equiv b_2 \pmod{2}$ are integers with $b_2 > b_1 > b_\ell(\alpha)$, then there is an integer $A_\ell(b_1, b_2, \alpha)$ such that for every nonnegative integer n we have*

$$p \left(\frac{\ell^{b_2 n} + 1}{24} \right) \equiv A_\ell(b_1, b_2, \alpha) p \left(\frac{\ell^{b_1 n} + 1}{24} \right) \pmod{\ell^\alpha}$$

where

$$b_\ell(\alpha) := 2 \left(\left\lfloor \frac{\ell - 1}{12} \right\rfloor + 2 \right) \alpha - 3.$$

Theorem 1.5. *If $5 \leq \ell \leq 31$ and $\alpha \geq 1$, then for $b \geq b_\ell(\alpha)$ we have that $P_\ell(b; 24z) \pmod{\ell^\alpha}$ is an eigenform of all of the weight $k_\ell(\alpha) - \frac{1}{2}$ Hecke operators on $\Gamma_0(576)$. Here*

$$P_\ell(b; z) := \sum_{n=0}^{\infty} p \left(\frac{\ell^b n + 1}{24} \right) q^{n/24}$$

Corollary 1.6. *Suppose that $5 \leq \ell \leq 31$ and that $\alpha \geq 1$. If $b \geq b_\ell(\alpha)$, then for every prime $p \geq 5$ there is an integer $\lambda_\ell(\alpha, p)$ such that for all n coprime to p we have*

$$p \left(\frac{\ell^b n p^3 + 1}{24} \right) \equiv \lambda_\ell(\alpha, p) p \left(\frac{\ell^b n p + 1}{24} \right) \pmod{\ell^\alpha}. \quad (11)$$

In [4], Boylan and Webb showed that, in Theorem 1.4, we can take $b_\ell(\alpha) = 2\alpha - 1$. They prove their results by showing that the generating functions

$$P_\ell(b; z) := \sum_{n=0}^{\infty} p \left(\frac{\ell^b n + 1}{24} \right) q^{n/24}$$

lie in a certain $\mathbb{Z}/\ell^m\mathbb{Z}$ module with finite rank, and for certain submodules the dimension is 1 when $5 \leq \ell \leq 31$.

In our proof of Conjecture 1.2, we mostly follow the proof of Atkin and O'Brien for Theorem 1.1 combined with the work of Newman [6].

2 Preliminaries

The full modular group is defined as

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}.$$

This group acts on the upper half-plane

$$\mathbb{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$$

by linear fractional (Möbius) transformations:

$$\gamma z = \frac{az + b}{cz + d}, \quad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

We will consider the subgroup $\Gamma_0(N)$ which is defined as

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\}$$

A *modular function* on $\Gamma_0(N)$ is a meromorphic function $f : \mathbb{H} \rightarrow \mathbb{C}$ satisfying:

1. For all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$,

$$f(\gamma z) = f(z).$$

2. The function f is meromorphic on the upper half-plane.
3. It might have poles at the cusps of $\Gamma_0(N)$.

A central role in our study is played by the Dedekind eta function, defined for $z \in \mathbb{H}$ by

$$\eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \quad \text{with } q = e^{2\pi iz}.$$

It is well known that $\eta(z)$ is holomorphic and non-vanishing on \mathbb{H} . Moreover, $\eta(z)$ satisfies the transformation law

$$\eta(\gamma z) = \nu_\eta(\gamma) (cz + d)^{1/2} \eta(z),$$

for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, where

$$\nu_\eta(\gamma) = \begin{cases} \left(\frac{d}{c}\right) e^{2\pi i((a+d)c - bd(c^2-1) - 3c)/24}, & \text{if } c \text{ is odd,} \\ \left(\frac{c}{d}\right) e^{2\pi i((a+d)c - bd(c^2-1) + 3d - 3 - 3cd)/24}, & \text{if } c \text{ is even,} \end{cases}$$

We say that $f \in M_{\lambda+1/2}^!(N, \psi \nu_\eta^r)$, where $\lambda \in \mathbb{Z}$, $N \in \mathbb{N}$, and ψ is a Dirichlet character modulo N , if

1. For all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ and all $z \in \mathbb{H}$,

$$f(\gamma z) = \psi(\gamma) \nu_\eta^r(\gamma) (cz + d)^{\lambda+1/2} f(z).$$

2. The function f is holomorphic on \mathbb{H} .
3. It might have poles at the cusps of $\Gamma_0(N)$.

The Hecke operators for $f \in M_{\lambda+1/2}^!(N, \psi\nu_\eta^r)$, $(r, 6) = 1$, and primes p are explicitly defined as ([1])

$$F|T_{p^2} = \sum_{n \equiv r(24)} \left(a(p^2n) + \left(\frac{-1}{p}\right)^{\frac{r-1}{2}} \left(\frac{12n}{p}\right) \psi(p)p^{\lambda-1}a(n) \right. \\ \left. + \psi^2(p)p^{2\lambda-1}a\left(\frac{n}{p^2}\right) \right) q^{\frac{n}{24}} \quad (12)$$

The Atkin U_p operator is defined as

$$f|U_p = \frac{1}{p} \sum_{j=0}^{p-1} f\left(\frac{\tau + 24j}{p}\right). \quad (13)$$

3 Results of Newman

Here, we will summarize the results of Newman ([6]) needed to prove the conjecture.

For distinct primes l and p , $p \geq 5$, and integers of opposite parity r and s , let

$$B(\tau) = \eta^r(\tau)\eta^s(l\tau), \quad B^*(\tau) = \eta^s(\tau)\eta^r(l\tau)$$

and

$$\Delta = \frac{(r+ls)(p^2-1)}{24}, \quad \Delta^* = \frac{(s+lr)(p^2-1)}{24}.$$

Recall that the Fricke involution sends τ to $-1/l\tau$. Then Newman showed the following

Lemma 3.1 (Newman). *Let*

$$G(\tau) = \frac{T_{p^2}(B(\tau))}{B(\tau)}, \quad G^*(\tau) = \frac{T_{p^2}(B^*(\tau))}{B^*(\tau)}.$$

Then $G(\tau)$ and $G^(\tau)$ are weakly holomorphic modular functions on $\Gamma_0(l)$, and*

$$G\left(-\frac{1}{l\tau}\right) = G^*(\tau)$$

The main result of Newman's paper is

Theorem 3.2 (Newman). *At $\tau = i\infty$, $G(\tau)$ has a pole of order $\lfloor \Delta/p^2 \rfloor$ at most if $\Delta \geq 0$, and a pole of order $-\Delta$ if $\Delta < 0$. At $\tau = 0$, $G(\tau)$ has a pole of order $\lfloor \Delta^*/p^2 \rfloor$ at most if $\Delta^* \geq 0$, and a pole of order $-\Delta^*$ if $\Delta^* < 0$.*

4 Proof of the Conjecture

Recall that

$$P(N) = \begin{cases} p(n) & \text{if } N = 24n - 1, \\ 0 & \text{if } N < -1 \text{ or } N \not\equiv -1 \pmod{24} \text{ or } N \text{ is nonintegral.} \end{cases}$$

Define

$$\varphi(\tau) = \frac{\eta(169\tau)}{\eta(\tau)} \tag{14}$$

$$g(\tau) = \left(\frac{\eta(13\tau)}{\eta(\tau)} \right)^2, \tag{15}$$

and

$$L_1 = \varphi|U_{13} = \eta(13\tau) \left(\frac{1}{\eta(\tau)} \Big| U_{13} \right) = \eta(13\tau) \sum_{N=0}^{\infty} P(13(24N + 11)) q^{N + \frac{11}{24}}. \tag{16}$$

In [3], Atkin and O'Brien showed that:

$$L_1 = \sum_{r=1}^7 k_{1r} g(\tau)^r, \tag{17}$$

giving

$$\frac{1}{\eta(\tau)} \Big| U_{13} = \sum_{N=0}^{\infty} P(13(24N + 11)) q^{N + \frac{11}{24}} = \sum_{r=1}^7 k_{1r} \frac{\eta(13\tau)^{2r-1}}{\eta(\tau)^{2r}}. \tag{18}$$

The valuations of k_{1r} are given in the following table

r	1	2	3	4	5	6	7
$\pi(k_{1r})$	0	1	2	3	4	5	5

Here $\pi(a)$, for integral a , is defined as:

$$13^{\pi(a)} \mid a, \quad 13^{\pi(a)+1} \nmid a.$$

We clearly have:

$$\pi(ab) = \pi(a) + \pi(b), \quad (19)$$

$$\pi(a+b) \geq \min(\pi(a), \pi(b)), \quad (20)$$

with equality unless $\pi(a) = \pi(b)$.

By ([2]),

$$\frac{1}{\eta(\tau)} \Big| U_{13} \in M_{-\frac{1}{2}} \left(13, \left(\frac{\cdot}{13} \right) \nu_{\eta}^{11} \right).$$

Direct calculation shows that

$$\frac{\eta(13\tau)^{2r-1}}{\eta(\tau)^{2r}} \in M_{-\frac{1}{2}} \left(13, \left(\frac{\cdot}{13} \right) \nu_{\eta}^{11} \right)$$

too. Therefore, we can apply T_{p^2} to both sides of (18)

$$\frac{1}{\eta(\tau)} \Big| U_{13} \Big| T_{p^2} = \sum_{r=1}^7 k_{1r} \left(\frac{\eta(13\tau)^{2r-1}}{\eta(\tau)^{2r}} \Big| T_{p^2} \right). \quad (21)$$

Next, let

$$\omega_r = \frac{\eta(13\tau)^{2r-1}}{\eta(\tau)^{2r}}, \quad \omega_r^* = \frac{\eta(\tau)^{2r-1}}{\eta(13\tau)^{2r}},$$

$$\Delta_r = \frac{(24r-13)(p^2-1)}{24}, \quad \Delta_r^* = -\frac{(24r+1)(p^2-1)}{24}.$$

and

$$G_r(\tau) = \frac{\omega_r \Big| T_{p^2}}{\omega_r}, \quad (22)$$

$$G_r^*(\tau) = \frac{\omega_r^* \Big| T_{p^2}}{\omega_r^*}. \quad (23)$$

Then, using Newman's results, $G_r(\tau)$ is a weakly holomorphic modular function on $\Gamma_0(13)$, G_r^* has a pole of order $-\Delta_r^*$ at $i\infty$, G_r has a pole of order $\lfloor \Delta_r/p^2 \rfloor$ at $i\infty$, and

$$G_r^* \left(-\frac{1}{13\tau} \right) = G_r(\tau). \quad (24)$$

This means that

$$G_r^*(\tau) - \sum_{i=1}^{-\Delta_r^*} \delta_{r,i} g(\tau)^{-i}$$

has no poles at $i\infty$ ($\delta_{r,i} \in \mathbb{Z}$). Using (24) and transformation properties of $\eta(\tau)$, we can rewrite that as

$$G_r^* \left(-\frac{1}{13\tau} \right) - \sum_{i=1}^{-\Delta_r^*} \delta_{r,i} g \left(-\frac{1}{13\tau} \right)^{-i} = G_r(\tau) - \sum_{i=1}^{-\Delta_r^*} \delta_{r,i} 13^i g(\tau)^i,$$

which does not have poles at 0. Since G_r has a pole of order $\lfloor \Delta_r/p^2 \rfloor$ at $i\infty$, and $\lfloor \Delta_r/p^2 \rfloor = r - 1$ for $1 \leq r \leq 7$, we have

$$G_r(\tau) - \sum_{i=1}^{-\Delta_r^*} \delta_{r,i} 13^i g(\tau)^i - \sum_{i=1}^{r-1} \delta_{r,-i} g(\tau)^{-i}$$

has no poles and must be a constant for $1 \leq r \leq 7$. Hence in general one writes:

$$G_r(\tau) = \sum_{i=1}^{r-1} \delta_{r,-i} g(\tau)^{-i} + \delta_{r,0} + \sum_{i=1}^{-\Delta_r^*} \delta_{r,i} 13^i g(\tau)^i.$$

Using (22), we will get

$$\omega_r|T_{p^2} = \omega_r(\tau) \left(\sum_{i=1}^{r-1} \delta_{r,-i} g(\tau)^{-i} + \delta_{r,0} + \sum_{i=1}^{-\Delta_r^*} \delta_{r,i} 13^i g(\tau)^i \right), \quad (25)$$

or, equivalently,

$$\omega_r|T_{p^2} = \sum_{i=1}^{-\Delta_r^*+r} h_{r,i} \omega_i. \quad (26)$$

One also sees conditions on $\pi(h_{r,i})$:

$$\pi(h_{r,i}) \geq 0 \quad \text{for } 1 \leq i \leq r, \quad \pi(h_{r,i}) \geq i - r \quad \text{for } i > r.$$

Thus,

$$\frac{1}{\eta(\tau)} \Big| U_{13} \Big|_{T_{p^2}} = \sum_{r=1}^7 k_{1r} \left(\frac{\eta(13\tau)^{2r-1}}{\eta(\tau)^{2r}} \Big|_{T_{p^2}} \right) = \sum_{r=1}^7 k_{1r} \sum_{j=1}^{-\Delta_r^*+r} h_{r,j} \omega_j, \quad (27)$$

or, after multiplying by $\eta(13\tau)$ and simplifying, we will get

$$\begin{aligned} \eta(13\tau) \left(\frac{1}{\eta(\tau)} \Big| U_{13} \Big|_{T_{p^2}} \right) &= \eta(13\tau) \sum_{N=0}^{\infty} \left(p^3 P(13p^2(24N+11)) + \right. \\ &+ p \left(\frac{-3}{p} \right) \left(\frac{13(24N+11)}{p} \right) P(13(24N+11)) + \\ &\left. + P \left(\frac{13(24N+11)}{p^2} \right) \right) q^{N+\frac{11}{24}} = \sum_{r=1}^7 k_{1r} \sum_{j=1}^{-\Delta_r^*+r} h_{r,j} g^j = \sum_{j \geq 1} u_{1,j} g^j \end{aligned}$$

Let

$$\begin{aligned} H_{2\alpha-1} &= \eta(13\tau) \left(\frac{1}{\eta(\tau)} \Big| U_{13}^{2\alpha-1} \Big|_{T_{p^2}} \right) = \eta(13\tau) \sum_{N=0}^{\infty} \left(p^3 P(13^{2\alpha-1} p^2 (24N+11)) \right. \\ &+ p \left(\frac{-3}{p} \right) \left(\frac{13^{2\alpha-1} (24N+11)}{p} \right) P(13^{2\alpha-1} (24N+11)) \\ &\left. + P \left(\frac{13^{2\alpha-1} (24N+11)}{p^2} \right) \right) q^{N+\frac{11}{24}} \end{aligned} \quad (28)$$

$$\begin{aligned} H_{2\alpha} &= \eta(\tau) \left(\frac{1}{\eta(\tau)} \Big| U_{13}^{2\alpha} \Big|_{T_{p^2}} \right) = \eta(13\tau) \sum_{N=0}^{\infty} \left(p^3 P(13^{2\alpha} p^2 (24N+23)) \right. \\ &+ p \left(\frac{-3}{p} \right) \left(\frac{13^{2\alpha} (24N+23)}{p} \right) P(13^{2\alpha} (24N+23)) \\ &\left. + P \left(\frac{13^{2\alpha} (24N+23)}{p^2} \right) \right) q^{N+\frac{23}{24}} \end{aligned} \quad (29)$$

$$L_{2\alpha-1} = \eta(13\tau) \left(\frac{1}{\eta(\tau)} \Big| U_{13}^{2\alpha-1} \Big|_{T_{p^2}} \right) = \eta(13\tau) \sum_{N=0}^{\infty} P(13^{2\alpha-1} (24N+11)) q^{N+\frac{11}{24}} \quad (30)$$

$$L_{2\alpha} = \eta(\tau) \left(\frac{1}{\eta(\tau)} \Big| U_{13}^{2\alpha} \Big|_{T_{p^2}} \right) = \eta(\tau) \sum_{N=0}^{\infty} P(13^{2\alpha} (24N+23)) q^{N+\frac{23}{24}} \quad (31)$$

In [3], Atkin and O'Brien showed that

$$g^k|U_{13} = \sum_r c_{k,r} g^r, \quad (32)$$

$$\varphi g^k|U_{13} = \sum_r d_{k,r} g^r, \quad (33)$$

and therefore

$$L_{2\alpha-1} = \varphi L_{2\alpha-2}|U_{13} = \sum_r k_{\alpha,r} g^r \quad (34)$$

$$L_{2\alpha} = L_{2\alpha-1}|U_{13} = \sum_r \ell_{\alpha,r} g^r. \quad (35)$$

Similarly, one can show that

$$H_{2\alpha-1} = \varphi H_{2\alpha-2}|U_{13} = \sum_r u_{\alpha,r} g^r, \quad (36)$$

$$H_{2\alpha} = H_{2\alpha-1}|U_{13} = \sum_r v_{\alpha,r} g^r. \quad (37)$$

Note that

$$l_{\alpha r} = \sum_j k_{\alpha j} c_{jr}, \quad (38)$$

$$k_{\alpha+1,r} = \sum_j l_{\alpha j} d_{jr}. \quad (39)$$

Lemma 4.1 (Atkin - O'Brien).

$$\begin{aligned} \pi(k_{\alpha,r}) &\geq \left\lfloor \frac{13r-9}{14} \right\rfloor, & \pi(\ell_{\alpha,r}) &\geq \left\lfloor \frac{13r-2}{14} \right\rfloor, \\ \pi(k_{\alpha,1}) &= 0, & \pi(\ell_{\alpha,1}) &= 0, \\ \pi(c_{k,r}) &\geq \left\lfloor \frac{13r-k-1}{14} \right\rfloor, & \pi(d_{k,r}) &\geq \left\lfloor \frac{13r-k-8}{14} \right\rfloor. \end{aligned}$$

Lemma 4.2.

$$\pi(u_{\alpha,r}) \geq \left\lfloor \frac{13r-9}{14} \right\rfloor, \quad \pi(v_{\alpha,r}) \geq \left\lfloor \frac{13r-2}{14} \right\rfloor.$$

Proof. For $\alpha = 1$, we have

$$\begin{aligned} H_1 &= \sum_{j \geq 1} u_{1,j} g^j = \sum_{r=1}^7 k_{1,r} \sum_{j=1}^{-\Delta_r^*+r} h_{r,j} g^j \\ &= k_{1,1} \sum_j h_{1,j} g^j + k_{1,2} \sum_j h_{2,j} g^j + \cdots + k_{1,7} \sum_j h_{7,j} g^j. \end{aligned}$$

Recall that

$$\begin{aligned} \pi(k_{1,1}) &= 0, \pi(k_{1,2}) = 1, \pi(k_{1,3}) = 2, \pi(k_{1,4}) = 3, \pi(k_{1,5}) = 4, \pi(k_{1,6}) = 5, \\ \pi(k_{1,7}) &= 5, \end{aligned}$$

and

$$\pi(h_{r,j}) \geq 0 \quad \text{for } 1 \leq j \leq r, \quad \pi(h_{r,j}) \geq j - r \quad \text{for } j > r.$$

Hence,

$$\pi(k_{1,r} h_{r,j}) \geq j - 1 \geq \left\lfloor \frac{13j - 9}{14} \right\rfloor, \quad 0 \leq r \leq 6,$$

and

$$\pi(k_{1,7} h_{7,j}) \geq \begin{cases} 5 & \text{for } j = 1, 2, \dots, 7 \\ j - 2 & \text{for } j \geq 8 \end{cases} \geq \left\lfloor \frac{13j - 9}{14} \right\rfloor.$$

Therefore,

$$\pi(u_{1,j}) \geq \min_r \{\pi(k_{1,r} h_{r,j})\} \geq \left\lfloor \frac{13j - 9}{14} \right\rfloor.$$

Next, consider

$$\begin{aligned} H_{2\alpha} &= \sum_{j \geq 1} v_{\alpha,j} g^j = H_{2\alpha-1} | U_{13} = \sum_{r \geq 1} u_{\alpha,r} (g^r | U_{13}) \\ &= \sum_{r \geq 1} u_{\alpha,r} \sum_{j \geq 1} c_{r,j} g^j = \sum_{j \geq 1} g^j \sum_{r \geq 1} u_{\alpha,r} c_{r,j}. \end{aligned}$$

Using the induction hypothesis and Lemma 4.1

$$\pi(u_{\alpha,r}) \geq \left\lfloor \frac{13r - 9}{14} \right\rfloor, \quad \pi(c_{r,j}) \geq \left\lfloor \frac{13j - r - 1}{14} \right\rfloor,$$

we get

$$\begin{aligned}\pi\left(\sum_{r \geq 1} u_{\alpha,r} c_{r,j}\right) &\geq \min_r \{\pi(u_{\alpha,r}) + \pi(c_{r,j})\} \\ &\geq \min_r \left\{ \left\lfloor \frac{13r-9}{14} \right\rfloor + \left\lfloor \frac{13j-r-1}{14} \right\rfloor \right\} = \left\lfloor \frac{13j-2}{14} \right\rfloor,\end{aligned}$$

since the minimum occurs at either $r = 1$ or $r = 2$, but in fact at $r = 1$. Hence,

$$\pi(v_{\alpha,r}) \geq \left\lfloor \frac{13r-2}{14} \right\rfloor.$$

Next,

$$\begin{aligned}H_{2\alpha+1} &= \sum_{j \geq 1} u_{\alpha+1,j} g^j = \varphi H_{2\alpha} | U_{13} = \sum_{r \geq 1} v_{\alpha,r} \left(\varphi g^r | U_{13} \right) \\ &= \sum_{r \geq 1} v_{\alpha,r} \sum_{j \geq 1} d_{r,j} g^j = \sum_{j \geq 1} g^j \sum_{r \geq 1} v_{\alpha,r} d_{r,j}.\end{aligned}$$

From Lemma 1 and the previous step,

$$\pi(v_{\alpha,r}) \geq \left\lfloor \frac{13r-2}{14} \right\rfloor, \quad \pi(d_{r,j}) \geq \left\lfloor \frac{13j-r-8}{14} \right\rfloor.$$

Thus,

$$\begin{aligned}\pi\left(\sum_{r \geq 1} v_{\alpha,r} d_{r,j}\right) &\geq \min_r \{\pi(v_{\alpha,r}) + \pi(d_{r,j})\} \\ &\geq \min_r \left\{ \left\lfloor \frac{13r-2}{14} \right\rfloor + \left\lfloor \frac{13j-r-8}{14} \right\rfloor \right\} = \left\lfloor \frac{13j-9}{14} \right\rfloor,\end{aligned}$$

since the minimum again occurs at $r = 1$. Hence,

$$\pi(u_{\alpha,r}) \geq \left\lfloor \frac{13r-9}{14} \right\rfloor.$$

This completes the proof of the lemma. \square

Now we are ready to prove the conjecture:

Conjecture 1.2. *Let $\alpha \geq 1$, and $p \neq 13$ be a prime ≥ 5 . Then there exists a constant $k = k(p, \alpha)$ such that for all N ,*

$$P(p^2 \cdot 13^\alpha N) - \left\{ k - \left(\frac{-3 \cdot 13^\alpha N}{p} \right) p^{-2} \right\} P(13^\alpha N) + p^{-3} P\left(\frac{13^\alpha N}{p^2}\right) \equiv 0 \pmod{13^\alpha},$$

where $\left(\frac{a}{b}\right)$ is the Jacobi symbol. Equivalently,

$$\left(\frac{1}{\eta(\tau)} \middle| U^\alpha \middle| T_{p^2} \right) \equiv k \left(\frac{1}{\eta(\tau)} \middle| U^\alpha \right) \pmod{13^\alpha}. \quad (40)$$

This is equivalent to

$$H_{2\alpha-1} \equiv \beta_\alpha L_{2\alpha-1} \pmod{13^{2\alpha-1}} \implies u_{\alpha,r} \equiv \beta_\alpha k_{\alpha,r} \pmod{13^{2\alpha-1}}.$$

and

$$H_{2\alpha} \equiv e_\alpha L_{2\alpha} \pmod{13^{2\alpha}} \implies v_{\alpha,r} \equiv e_\alpha \ell_{\alpha,r} \pmod{13^{2\alpha}}.$$

Let

$$\mu_{st}^\alpha = u_{\alpha,s} k_{\alpha,t} - u_{\alpha,t} k_{\alpha,s}, \quad \gamma_{st}^\alpha = v_{\alpha,s} \ell_{\alpha,t} - v_{\alpha,t} \ell_{\alpha,s}.$$

We want to show

$$\pi(\mu_{st}^\alpha) \geq 2\alpha - 1 \quad \text{for all } s, t \geq 1, \quad \pi(\gamma_{st}^\alpha) \geq 2\alpha \quad \text{for all } s, t \geq 1,$$

since if we set $s = r$ and $t = 1$, then

$$\mu_{r1}^\alpha = u_{\alpha,r} k_{\alpha,1} - u_{\alpha,1} k_{\alpha,r} \equiv 0 \pmod{13^{2\alpha-1}},$$

$$\gamma_{r1}^\alpha = v_{\alpha,r} \ell_{\alpha,1} - v_{\alpha,1} \ell_{\alpha,r} \equiv 0 \pmod{13^{2\alpha}},$$

and

$$u_{\alpha,r} \equiv \frac{u_{\alpha,1}}{k_{\alpha,1}} k_{\alpha,r} \pmod{13^{2\alpha-1}}, \quad v_{\alpha,r} \equiv \frac{v_{\alpha,1}}{\ell_{\alpha,1}} \ell_{\alpha,r} \pmod{13^{2\alpha}}.$$

Since $k_{\alpha,1} \not\equiv 0 \pmod{13}$ and $\ell_{\alpha,1} \not\equiv 0 \pmod{13}$ by Lemma 4.1, one obtains the claimed congruence.

Theorem 4.4. *We have the following lower bounds on valuations:*

$$\pi(\mu_{s,t}^\alpha) \geq 2\alpha - 1 + \left\lfloor \frac{13(s+t) - 46}{14} \right\rfloor, \quad \text{if } s+t > 3, \quad (41)$$

$$\pi(\mu_{s,t}^\alpha) \geq 2\alpha - 1, \quad \text{if } s+t = 3, \quad (42)$$

$$\pi(\gamma_{s,t}^\alpha) \geq 2\alpha + \left\lfloor \frac{13(s+t) - 33}{14} \right\rfloor. \quad (43)$$

Proof. First, note that when $s+t=2$,

$$\mu_{1,1}^\alpha = 0 \quad \text{and} \quad \gamma_{1,1}^\alpha = 0.$$

So, in what follows, we will assume that $s+t \geq 3$.

For $\alpha = 1$, we have

$$\mu_{s,t}^1 = u_{1,s}k_{1,t} - u_{1,t}k_{1,s}.$$

Recall the known valuations:

$$\pi(u_j) \geq \left\lfloor \frac{13j-9}{14} \right\rfloor, \quad \pi(k_j) \geq \left\lfloor \frac{13j-9}{14} \right\rfloor.$$

Hence

$$\begin{aligned} \pi(\mu_{st}^1) &\geq \min\{\pi(u_{1s}) + \pi(k_{1t}), \pi(u_{1t}) + \pi(k_{1s})\} = \\ &= \min\left\{\left\lfloor \frac{13s-9}{14} \right\rfloor + \left\lfloor \frac{13t-9}{14} \right\rfloor, \left\lfloor \frac{13t-9}{14} \right\rfloor + \left\lfloor \frac{13s-9}{14} \right\rfloor\right\} \geq \\ &\geq \left\lfloor \frac{13(s+t) - 18 - 13}{14} \right\rfloor = 1 + \left\lfloor \frac{13(s+t) - 45}{14} \right\rfloor \geq \\ &\geq 1 + \left\lfloor \frac{13(s+t) - 46}{14} \right\rfloor \quad \text{if } s+t > 3. \end{aligned}$$

If $(s+t) = 3$, then either $s=1, t=2$ or $s=2, t=1$, and we get the result by direct calculation.

Assume the statement holds for all $\mu_{s,t}^k$, $k \leq \alpha$. Next, consider

$$\begin{aligned} \gamma_{st}^\alpha &= v_{\alpha s} \ell_{\alpha t} - v_{\alpha t} \ell_{\alpha s} = \left(\sum u_{\alpha i} c_{is} \right) \left(\sum k_{\alpha j} c_{jt} \right) - \left(\sum u_{\alpha i} c_{it} \right) \left(\sum k_{\alpha j} c_{js} \right) \\ &= \sum c_{is} c_{jt} (u_{\alpha i} k_{\alpha j} - u_{\alpha j} k_{\alpha i}) = \sum c_{is} c_{jt} \mu_{ij}^\alpha \end{aligned}$$

Then

$$\begin{aligned}\pi(\gamma_{st}^\alpha) &\geq \min_{i,j} \left\{ \left\lfloor \frac{13s-i-1}{14} \right\rfloor + \left\lfloor \frac{13t-j-1}{14} \right\rfloor + 2\alpha - 1 + \left\lfloor \frac{13(i+j)-46}{14} \right\rfloor \right\} \geq \\ &\geq \min_{i,j} \left\{ \left\lfloor \frac{13(s+t)-(i+j)-15}{14} \right\rfloor + 2\alpha - 1 + \left\lfloor \frac{13(i+j)-46}{14} \right\rfloor \right\}\end{aligned}$$

if $i+j > 3$ and

$$\begin{aligned}\pi(\gamma_{st}^\alpha) &\geq \left\lfloor \frac{13(s+t)-3-15}{14} \right\rfloor + 2\alpha - 1 = \\ &= 2\alpha + \left\lfloor \frac{13(s+t)-32}{14} \right\rfloor \quad \text{if } i+j = 3\end{aligned}$$

The total minimum is obtained if $i+j = 4$. Therefore,

$$\pi(\gamma_{st}^\alpha) \geq 2\alpha + \left\lfloor \frac{13(s+t)-33}{14} \right\rfloor$$

To complete our induction, we need to prove the bound for $\mu_{s,t}^{\alpha+1}$. Since

$$\begin{aligned}\mu_{st}^{\alpha+1} &= u_{\alpha+1,s}k_{\alpha+1,t} - u_{\alpha+1,t}k_{\alpha+1,s} = \left(\sum v_{\alpha i}d_{is} \right) \left(\sum \ell_{\alpha j}d_{jt} \right) \\ &\quad - \left(\sum v_{\alpha i}d_{it} \right) \left(\sum \ell_{\alpha j}d_{js} \right) = \sum d_{is}d_{jt}\gamma_{ij}^\alpha,\end{aligned}$$

we have

$$\begin{aligned}\pi(\mu_{st}^{\alpha+1}) &\geq \min_{i,j} \left\{ \left\lfloor \frac{13s-i-8}{14} \right\rfloor + \left\lfloor \frac{13t-j-8}{14} \right\rfloor + 2\alpha + \left\lfloor \frac{13(i+j)-33}{14} \right\rfloor \right\} \geq \\ &\geq \min \left\{ \left\lfloor \frac{13(s+t)-(i+j)-29}{14} \right\rfloor + 2\alpha + \left\lfloor \frac{13(i+j)-33}{14} \right\rfloor \right\} = \\ &= \left\lfloor \frac{13(s+t)-3-29}{14} \right\rfloor + 2\alpha + \left\lfloor \frac{13 \cdot 3 - 33}{14} \right\rfloor = \\ &= 2\alpha + 1 + \left\lfloor \frac{13(s+t)-46}{14} \right\rfloor\end{aligned}$$

for $s+t > 3$ since the minimum is reached when $i+j = 3$. For $s+t = 3$,

$i + j > 3$ we get

$$\begin{aligned}
\pi(\mu_{st}^{\alpha+1}) &\geq \min_{i,j} \left\{ \left\lfloor \frac{13s - i - 8}{14} \right\rfloor + \left\lfloor \frac{13t - j - 8}{14} \right\rfloor + 2\alpha + \left\lfloor \frac{13(i + j) - 33}{14} \right\rfloor \right\} \geq \\
&\geq \min \left\{ \left\lfloor \frac{13(s + t) - (i + j) - 29}{14} \right\rfloor + 2\alpha + \left\lfloor \frac{13(i + j) - 33}{14} \right\rfloor \right\} = \\
&= \min \left\{ \left\lfloor \frac{10 - (i + j)}{14} \right\rfloor + 2\alpha + \left\lfloor \frac{13(i + j) - 33}{14} \right\rfloor \right\} = \\
&= 2\alpha + 1.
\end{aligned}$$

Finally, if $s + t = 3$ and $i + j = 3$, then

$$\pi(\mu_{st}^{\alpha+1}) \geq 2\alpha + 1$$

by direct calculation. □

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