A NEW APPROACH TO THE DYSON RANK CONJECTURES

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Dedicated to the memory of Richard Askey, a great friend and mentor

ABSTRACT. In 1944 Dyson defined the rank of a partition as the largest part minus the number of parts, and conjectured that the residue of the rank mod 5 divides the partitions of $5n + 4$ into five equal classes. This gave a combinatorial explanation of Ramanujan's famous partition congruence mod 5. He made an analogous conjecture for the rank mod 7 and the partitions of $7n + 5$. In 1954 Atkin and Swinnerton-Dyer proved Dyson's rank conjectures by constructing several Lambert-series identities basically using the theory of elliptic functions. In 2016 the author gave another proof using the theory of weak harmonic Maass forms. In this paper we describe a new and more elementary approach using Hecke-Rogers series.

1. SOME GUESSES IN THE THEORY OF PARTITIONS

In 1944, Freeman Dyson [11], as an undergraduate at Cambridge, wrote an article with the title of this section, in which he made a number of conjectures related to Ramanujan's famous partition congruences. Let $p(n)$ denote the number of partitions of n. Ramanujan discovered and later proved three beautiful congruences for the partition function, namely

$$p(5n+4) \equiv 0 \pmod{5}, \tag{1.1}$$

$$p(7n+5) \equiv 0 \pmod{7}, \tag{1.2}$$

$$p(11n+6) \equiv 0 \pmod{11}. \tag{1.3}$$

Dyson went on to remark that although there at least four different proofs of (1.1) and (1.2), it would be more satisfying to have a direct proof of (1.1). By this, he supposed whether there was some natural way of dividing the partitions of $5n + 4$ into five equally numerous classes. He went on to define the *rank* of a partition as the largest part minus the number of parts, and conjectured that the residue of the rank mod 5 does the job of dividing the partitions of $5n + 4$ into five equal classes. He also conjectured that the residue of the rank

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Date: December 11, 2020.

2020 *Mathematics Subject Classification.* 05A17, 11F33, 11P81, 11P83, 11P84, 33D15.

Key words and phrases. Dyson's rank conjectures, Ramanujan partition congruences, Hecke-Rogers series.

The author was supported in part by a grant from the Simon's Foundation (#318714). A preliminary version of this paper was presented October 1, 2020 at the Number Theory Seminar, St. Petersburg State University and Euler International Mathematical Institute, Russia. It was also presented October 3, 2020 at the Special Session on q-Series and Related Areas in Combinatorics and Number Theory, A.M.S. Fall Eastern Sectional Meeting.

mod 7 in a similar way divides the partitions of $7n+5$ into seven equal classes thus explaining (1.2).

More explicitly, Dyson denoted by $N(m,n)$, the number of partitions of n with rank m, and let $N(m,t,n)$ denote the number of partitions of n with rank congruent to m mod t. We restate

Dyson's Rank Conjectures 1.1 (1944)**.** *For all nonnegative integers n,*

$$N(0,5,5n+4) = N(1,5,5n+4) = \cdots = N(4,5,5n+4) = \tfrac{1}{5}p(5n+4), \tag{1.4}$$

$$N(0,7,7n+5) = N(1,7,7n+5) = \cdots = N(6,7,7n+5) = \tfrac{1}{7}p(7n+5). \tag{1.5}$$

The corresponding conjecture with modulus 11 is definitely false. Towards the end of his article, Dyson conjectured that there is a hypothetical statistic he dubbed the *crank*, whose residue mod 11 should divide the partitions of $11n+6$ into eleven classes thus explaining (1.3). The later discovery of the crank is another story [6].

The Dyson Rank Conjectures (1.4) and (1.5) were first proved by Atkin and Swinnerton-Dyer [7] in 1954. Atkin and Swinnerton-Dyer's proof involved proving many theta-function and generalized Lambert (or Appell-Lerch) series identities using basically elliptic function techniques. Their method also involved finding identities for the generating functions of $N(k,5,5n+r)$ for all the residues $r=0$, 1, 2, 3 and 4. We quote from their paper [7, p.84]:

> It is noteworthy that we have to obtain at the same time all the results stated in these theorems— we cannot simplify the working so as to merely to obtain Dyson's identities.

Only recently have new methods for approaching the Dyson Rank Conjectures been found. In 2017 Hickerson and Mortenson [17] used their theory of Appell-Lerch series to obtain results for the Dyson rank function including the Dyson Rank Conjectures. In 2019 the author [13] showed how the theory of harmonic Maass forms can be used to prove the Dyson Rank Conjectures and much more. In this paper we describe a new and more elementary method for proving Dyson's Rank Conjectures. The method only relies on identities for Hecke-Rogers series. We describe these series identities in the next section. We are able to obtain (1.4) for partitions of $5n+4$ without having to deal with the partitions of $5n+r$ for the other residues $r=0$, 1, 2 and 3.

2. Hecke-Rogers series

The main theorem of this section is Theorem 2.4, which contains four two-variable generalized Hecke-Rogers series identities for the Dyson rank generating function. Only two of these identities are needed to prove Dyson's mod 5 rank conjecture (1.4). Regarding the other two identities, we will use one identity to obtain Ramanujan's 5-dissection rank identity from the Lost Notebook [21, p.20]. The last identity is connected with Dyson's mod 7 rank conjecture.

One of these Hecke-Rogers identities is known. Two follow from a general result of Bradley-Thrush [9]. Our proof of the remaining identity uses results of Hickerson and Mortenson [16] for Appell-Lerch series.

Following [2, p.84] a *Hecke-Rogers series* has the form

$$\sum_{(n,m)\in D} (\pm 1)^{f(n,m)} q^{Q(n,m)+L(n,m)}$$

where Q is an indefinite binary quadratic form, L is a linear form and D is a subset of $\mathbb{Z}^2$ for which $Q(n,m) \geq 0$. L. J. Rogers [22, p.323] found

$$\prod_{n=1}^{\infty}(1-q^n)^2 = \sum_{n=-\infty}^{\infty} \sum_{m=-\lfloor n/2\rfloor}^{\lfloor n/2\rfloor} (-1)^{n+m} q^{(n^2-3m^2)/2+(n+m)/2} \tag{2.1}$$

The first systematic study of such series was done by Hecke [15] who independently obtained Rogers identity. Identities of this type arose in Kac and Petersen's [19] work on character formulas for infinite dimensional Lie algebras and string functions. Andrews [2] showed how identities of this type can be derived using his constant term method.

We have the following two-variable generating function for the rank [14, Eq.(7.2),p.66]: Then

(2.2)

$$R(z;q) := \sum_{n=0}^{\infty}\sum_{m} N(m,n) z^m q^n = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(zq;q)_n (z^{-1}q;q)_n} = \frac{(1-z)}{(q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}}{(1-zq^n)},$$

where the last equality follows easily from [14, Eq.(7.10),p.68]. Here we are using the usual q-notation:

$$(a)_n = (a;q)_n := (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1}),$$
$$(a)_\infty = (a;q)_\infty = \lim_{n\to\infty}(a;q)_n = \prod_{n=1}^{\infty}(1-aq^{n-1}),$$

provided $|q| < 1$, and recalling that $N(m,n)$ is the number of partitions of n with rank m. We will often use the Jacobi triple product identity [1, Theorem 3.4, p.461] for the theta-function $j(z;q)$:

$$j(z;q) := (z;q)_\infty (z^{-1}q;q)_\infty (q;q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n z^n q^{n(n-1)/2}. \tag{2.3}$$

Also we use the following notation for theta-type q-products:

$$J_{b,a} := j(q^a;q^b), \quad \text{and} \quad J_b := (q^b;q^b)_\infty. \tag{2.4}$$

We need the following general result of Bradley-Thrush [9].

Theorem 2.1 (Bradley-Thrush [9, Theorem 7.5]). *Let p, q, x and y be non-zero complex numbers such that $|p|$, $|q| < 1$ and let k be a positive integer such that $|pq^{-k^2}| < 1$. Then*

$$j(y;q)\sum_{n=-\infty}^{\infty}\frac{(-1)^n p^{n(n+1)/2}x^n}{1-yq^{kn}} = \sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty}(-1)^{m+n}q^{(n-m)(n-m+1)/2}j(q^{-kn}x^{-1};p)y^m. \tag{2.5}$$

We also need some notation and results of Hickerson and Mortenson [16], [17]. First we give Hickerson and Mortenson's definitions of their functions $f_{a,b,c}(x,y,q)$, $m(x,q,z)$, $g_{a,b,c}(x,y,q,z_1,z_0)$ and $g(x,q)$:

$$f_{a,b,c}(x,y,q) := \sum_{\mathrm{sg}(r)=\mathrm{sg}(s)} \mathrm{sg}(r)(-1)^{r+s}x^r y^s q^{a\binom{r}{2}+brs+c\binom{s}{2}}. \tag{2.6}$$

where $\mathrm{sg}(r) := 1$ for $r \geq 0$ and $\mathrm{sg}(r) := -1$ for $r < 0$.

$$m(x,q,z) := \frac{1}{j(z;q)}\sum_{r=-\infty}^{\infty}\frac{(-1)^r q^{\binom{r}{2}}z^r}{1-q^{r-1}xz}. \tag{2.7}$$

$$\begin{aligned} &g_{a,b,c}(x,y,q,z_1,z_0) \\ &:= \sum_{t=0}^{a-1}(-y)^t q^{c\binom{t}{2}} j(q^{bt}x;q^a)\, m\left(-q^{a\binom{b+1}{2}-c\binom{a+1}{2}-t(b^2-ac)}\frac{(-y)^a}{(-x)^b}, q^{a(b^2-ac)}, z_0\right) \\ &\quad + \sum_{t=0}^{c-1}(-x)^t q^{a\binom{t}{2}} j(q^{bt}y;q^c)\, m\left(-q^{c\binom{b+1}{2}-a\binom{c+1}{2}-t(b^2-ac)}\frac{(-x)^c}{(-y)^b}, q^{c(b^2-ac)}, z_1\right). \end{aligned} \tag{2.8}$$

$$g(x,q) := x^{-1}\left(-1+\sum_{n=0}^{\infty}\frac{q^{n^2}}{(x;q)_{n+1}(q/x;q)_n}\right). \tag{2.9}$$

From (2.2) we see that $g(x,q)$ is related to the rank function $R(z;q)$ by

$$g(x,q) = x^{-1}\left(-1+\frac{1}{1-x}R(x;q)\right),$$

and

$$R(z;q) = (1-z)\left(1+zg(z,q)\right). \tag{2.10}$$

The function $g(z,q)$ is related the m-function by

$$g(z,q) = -z^{-2}m(z^{-3}q,q^3,z^2) - z^{-1}m(z^{-3}q^2,q^3,z^2). \tag{2.11}$$

See [17, Eq.(26a)].

We need

Theorem 2.2 (Hickerson and Mortenson [16, Theorem 1.6]). *Let n be a positive integer. For generic x, $y \in \mathbb{C}^*$*

$$f_{n,n+1,n}(x,y,q) = g_{n,n+1,n}(x,y,q,y^n/x^2,x^n/y^n).$$

Remark. By generic, Hickerson and Mortenson mean values that do not cause singularities at poles of Appell-Lerch series or quotients of theta-functions. Also $\mathbb{C}^*$ is the set of non-zero complex numbers.

We need some well-known properties of Appell-Lerch series [17, Proposition 3.1]

$$m(x,q,z) = m(x,q,qz), \tag{2.12}$$

$$m(x,q,z) = x^{-1}m(x^{-1},q,z^{-1}), \tag{2.13}$$

$$m(qx,q,z) = 1 - xm(x,q,z), \tag{2.14}$$

$$m(x,q,z_1) - m(x,q,z_0) = \frac{z_0 J_1^3 j(z_1/z_0;q) j(xz_0z_1;q)}{j(z_0;q)j(z_1;q)j(xz_0;q)j(xz_1;q)}, \tag{2.15}$$

for generic x, z, z_0, $z_1 \in \mathbb{C}^*$. The following is a well-known three term theta-function identity

$$\begin{aligned} &j(d;q)j(bc^{-1};q)j(abc;q)j(ad;q) - j(b;q)j(dc^{-1};q)j(acd;q)j(ab;q) \\ &\quad + bc^{-1}j(c;q)j(abd;q)j(ac;q)j(db^{-1};q) = 0. \end{aligned} \tag{2.16}$$

See [9, Theorem 4.1].

We will also need the following lemma.

Lemma 2.3.

$$\begin{aligned} &zj(z^{-1}q;q)m(q,q^3,z) + zj(z^{-2}q;q)m(z^3q,q^3,z^{-1}) \\ &\quad + z^2j(zq;q)m(q,q^3,z^{-1}) + z^2j(z^2q;q)m(z^{-3}q,q^3,z) \\ &= -j(z^2;q)m(z^{-3}q,q^3,z^2) + zj(z^2;q)m(z^3q,q^3,z^{-2}). \end{aligned} \tag{2.17}$$

Proof. We apply (2.15) three times to obtain

$$\begin{aligned} &zj(z^{-1}q;q)m(q,q^3,z) + z^2j(zq;q)m(q,q^3,z^{-1}) \\ &= z^2j(zq;q)\left(m(q,q^3,z^{-1}) - m(q,q^3,z)\right) \\ &= \frac{z^3j(z^{-2};q^3)j(q;q^3)j(zq;q)}{j(z;q^3)j(z^{-1};q^3)j(zq;q^3)j(z^{-1}q;q^3)}, \end{aligned}$$

$$\begin{aligned} &zj(z^{-2}q;q)m(z^3q,q^3,z^{-1}) - zj(z^2;q)m(z^3q,q^3,z^{-2}) \\ &= zj(z^2q,q)\left(m(z^3q;q^3,z^{-1}) - m(z^3q,q^3,z^{-2})\right) \\ &= \frac{j(z^2;q^3)j(z;q^3)j(q;q^3)}{zj(z^{-2},q^3)j(z^{-1};q^3)j(zq;q^3)j(z^2q;q^3)}, \end{aligned}$$

$$\begin{aligned} &j(z^2;q)m(z^{-3}q,q^3,z^2) + z^2j(z^2q;q)m(z^{-3}q,q^3,z) \\ &= j(z^2,q)\left(m(z^{-3}q,q^3,z^2) - m(z^{-3}q,q^3,z)\right) \\ &= \frac{zj(z^2;q)j(q,q^3)}{j(z^2;q^3)j(z^{-2}q;q^3)j(z^{-1}q;q^3)}. \end{aligned}$$

Thus we see that (2.17) is equivalent to showing that

$$\frac{z^3 j(z^{-2};q^3)j(q;q^3)j(zq;q)}{j(z;q^3)j(z^{-1};q^3)j(zq;q^3)j(z^{-1}q;q^3)} + \frac{j(z^2;q^3)j(z;q^3)j(q;q^3)}{zj(z^{-2};q^3)j(z^{-1};q^3)j(zq;q^3)j(z^2q;q^3)} + \frac{j(z^2;q)j(q;q^3)}{j(z^2;q^3)j(z^{-2}q;q^3)j(z^{-1}q;q^3)} = 0.$$

By rewriting each function on base q^3 we find that this identity is equivalent to

$$zj(zq^2;q^3)j(z^2q^2;q^3)j(z;q^3) - j(zq^2;q^3)j(zq;q^3)j(z^2;q^3) + j(z;q^3)j(zq;q^3)j(z^2q;q^3) = 0.$$

This identity is a special case of (2.16). This completes the proof of (2.17). ☐

Theorem 2.4.

(2.18)

$$\begin{aligned}(zq)_\infty(z^{-1}q)_\infty(q)_\infty R(z;q) &= \frac{1}{2}\sum_{n=0}^{\infty}\left(\sum_{j=0}^{[n/2]}(-1)^{n+j}(z^{n-3j}+z^{3j-n})q^{\frac{1}{2}(n^2-3j^2)+\frac{1}{2}(n-j)}\right.\\ &\qquad \left. + \sum_{j=1}^{[n/2]}(-1)^{n+j}(z^{n-3j+1}+z^{3j-n-1})q^{\frac{1}{2}(n^2-3j^2)+\frac{1}{2}(n+j)}\right),\end{aligned}$$

(2.19)

$$(zq)_\infty(z^{-1}q)_\infty(q)_\infty R(z;q^2) = \sum_{n=0}^{\infty}\sum_{j=-n}^{n}(-1)^j z^j q^{\frac{1}{2}n(3n+1)-j^2}(1-q^{2n+1})$$

(2.20)

$$(1+z)(z^2q;q)_\infty(z^{-2}q;q)_\infty(q;q)_\infty R(z;q) = \sum_{n=0}^{\infty}\sum_{j=-\lfloor n/2\rfloor}^{\lfloor n/2\rfloor}(-1)^{n+j}(z^{n+1}+z^{-n})q^{\frac{1}{2}(n^2-3j^2)+\frac{1}{2}(n-j)}$$

(2.21)

$$\begin{aligned}&(1+z)(z^2q^2;q^2)_\infty(z^{-2}q^2;q^2)_\infty(q^2;q^2)_\infty R(z;q)\\ &= \sum_{n=0}^{\infty}\sum_{j=0}^{2n}(-1)^n(z^{j+1}+z^{-j})q^{3n^2+2n-\frac{1}{2}j(j+1)} - \sum_{n=1}^{\infty}\sum_{j=0}^{2n-2}(-1)^n(z^{j+1}+z^{-j})q^{3n^2-2n-\frac{1}{2}j(j+1)}\end{aligned}$$

Remark. By letting $z = 1$ we see that identities (2.18) and (2.20) are z-analogs of (2.1). Similarly we find that (2.19) is z-analog of the identity [3, Eq.(7.2),p.66]

$$(q)_\infty^2(q;q^2)_\infty = \sum_{n=0}^{\infty}\sum_{j=-n}^{n}(-1)^j q^{\frac{1}{2}n(3n+1)-j^2}(1-q^{2n+1}).$$

Proof of (2.18). Equation (2.18) is [12, Eq. (1.15), p.269]. For a proof see [12, Section 3].

Proof of (2.19). We show how (2.19) follows from the following result of Bradley-Thrush's Theorem 2.1. In Equation (2.5) we let $k = 2$, $p = q^6$, $x = q^{-2}$, $y = z$ and noting that $|pq^{-k^2}| = |q|^2 < 1$ we find that

$$j(z;q)\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n(3n+1)}x^n}{1-zq^{2n}} = \sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty}(-1)^{m+n}q^{(n-m)(n-m+1)/2}j(q^{-2(n-1)};q^6)z^m.$$

Now

$$j(q^{-2(n-1)};q^6) = (-1)^{n+1}\left(\frac{n-1}{3}\right)q^{-n(n+1)/3}(q^2;q^2)_\infty,$$

which follows easily from Jacobi's triple product (2.3). See also [10, Eq.(4.8)] or [7, p.99]. By this and (2.2) we have

$$\begin{aligned}\frac{j(z;q)}{1-z}R(z;q^2) &= \sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty}(-1)^{m+1}\left(\frac{n-1}{3}\right)q^{m(m-1)/2-mn+n(n+1)/6}z^m\\ &= \sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty}(-1)^{m+1}\left(\frac{n-1}{3}\right)q^{(m^2+|m|)/2+n|m|+n(n+1)/6}z^m.\end{aligned}$$

For the last equation we used symmetry in z. Then in this last sum we replace n by $n-3|m|$ and use (2.3) to obtain

$$\begin{aligned}(zq)_\infty(z^{-1}q)_\infty(q)_\infty R(z;q^2) &= \sum_{m=-\infty}^{\infty}\sum_{n\geq 3|m|}(-1)^{m+1}\left(\frac{n-1}{3}\right)q^{n(n+1)/6-m^2}z^m\\ &= \sum_{n=0}^{\infty}\sum_{m=-\lfloor n/3\rfloor}^{\lfloor n/3\rfloor}(-1)^{m+1}\left(\frac{n-1}{3}\right)q^{n(n+1)/6-m^2}z^m.\end{aligned}$$

We find that the result (2.19) follows by replacing n by $3n+k$ where $k = -1$, 0.

Proof of (2.20). We prove (2.20) by rewriting the right-side in terms Hickerson and Mortenson's $f_{1,2,1}$ and then by using one of their theorems and some identities for Appell-Lerch series. From (2.6) we find that

$$f_{1,2,1}(z^{-1}q, z^{-2}q, q)$$

$$= \sum_{n=0}^{\infty} \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^{n+j} z^{-n} q^{\frac{1}{2}(n-3j^2)+\frac{1}{2}(n-j)} + \sum_{n=0}^{\infty} \sum_{j=-\lfloor n/2 \rfloor}^{-1} (-1)^{n+j} z^{n+1} q^{\frac{1}{2}(n-3j^2)+\frac{1}{2}(n-j)}.$$

Therefore

(2.22)

$$f_{1,2,1}(z^{-1}q, z^{-2}q, q) + z f_{1,2,1}(zq, z^2q, q) = \sum_{n=0}^{\infty} \sum_{j=-\lfloor n/2 \rfloor}^{\lfloor n/2 \rfloor} (-1)^{n+j} \left(z^{n+1} + z^{-n} \right) q^{\frac{1}{2}(n-3j^2)+\frac{1}{2}(n-j)}.$$

From Theorem 2.2 with $n = 1$ we have

(2.23)

$$\begin{aligned} & f_{1,2,1}(z^{-1}q, z^{-2}q, q) + z f_{1,2,1}(zq, z^2q, q) \\ & = g_{1,2,1}(z^{-1}q, z^{-2}q, q, z^{-1}, z) + z g_{1,2,1}(zq, z^2q, q, z, z^{-1}) \\ & = j(z^{-1}q, q) m(q, q^3, z) + j(z^{-2}q, q) m(z^3q, q^3, z^{-1}) \\ & \quad + z j(zq; q) m(q, q^3, z^{-1}) + z j(z^2q, q) m(z^{-3}q, q^3, z) \\ & = j(z^2; q) \left(m(z^3q, q^3, z^{-2}) - z^{-1} m(z^{-3}q, q^3, z^2) \right) \end{aligned}$$

by Lemma 2.3. By (2.10), (2.11), (2.13)–(2.14), (2.23) and (2.22) we have

$$\begin{aligned} & (1+z)(z^2q; q)_\infty (z^{-2}q; q)_\infty (q; q)_\infty R(z; q) = j(z^2; q) \frac{1}{1-z} R(z, q) \\ & = j(z^2; q)(1 + z\, g(z, q)) \\ & = j(z^2; q)(1 - z^{-1} m(z^{-3}q, q^3, z^2) - m(z^{-3}q^2, q^3, z^2)) \\ & = f_{1,2,1}(z^{-1}q, z^{-2}q, q) + z f_{1,2,1}(zq, z^2q, q) \\ & = \sum_{n=0}^{\infty} \sum_{j=-\lfloor n/2 \rfloor}^{\lfloor n/2 \rfloor} (-1)^{n+j} \left(z^{n+1} + z^{-n} \right) q^{\frac{1}{2}(n-3j^2)+\frac{1}{2}(n-j)}. \end{aligned}$$

Proof of (2.21). The proof of (2.21) is similar to that of (2.19). From (2.2) we have

$$\begin{aligned} & (1+z)(z^2q^2; q^2)_\infty (z^{-2}q^2; q^2)_\infty (q^2; q^2)_\infty R(z; q) = \frac{j(z^2; q^2)}{1-z} R(z; q) \\ & = \frac{j(z^2; q^2)}{(q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\frac{1}{2}n(3n+1)}}{(1 - zq^n)} \\ & = \frac{j(z^2; q^2)}{(q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\frac{1}{2}n(3n+1)}(1 + zq^n)}{(1 - z^2q^{2n})}. \end{aligned}$$

Next we make two applications of Theorem 2.1, with $k = 1$, $p = q^3$, $q \mapsto q^2$, $y = z^2$ and $x = q^{-1}$ then $x = 1$, so that

$$(1+z)(z^2q^2; q^2)_\infty (z^{-2}q^2; q^2)_\infty (q^2; q^2)_\infty R(z; q)$$

$$= \frac{1}{(q)_\infty}\left(\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty}(-1)^{m+n}q^{(n-m)(n-m+1)}j(q^{-2n+1};q^3)z^{2m}\right.$$
$$\left.+\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty}(-1)^{m+n}q^{(n-m)(n-m+1)}j(q^{-2n};q^3)z^{2m+1}\right).$$

Now

$$j(q^n;q^3) = (-1)^{n+1}\left(\frac{n}{3}\right)q^{-\frac{1}{6}(n-1)(n-2)}(q)_\infty,$$

which follows from Jacobi's triple product (2.3). Then after some simplification and utilising symmetry in z we find that

$$(1+z)(z^2q^2;q^2)_\infty(z^{-2}q^2;q^2)_\infty(q^2;q^2)_\infty R(z;q)$$
$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\lfloor n/3\rfloor}(-1)^n\left(\frac{n+1}{3}\right)q^{\frac{1}{3}n(n+2)-m(2m+1)}(z^{-2m}+z^{2m+1})$$
$$+\sum_{n=0}^{\infty}\sum_{m=0}^{\lfloor n/3\rfloor}(-1)^n\left(\frac{n}{3}\right)q^{\frac{1}{3}(n+1)(n+5)-m(2m+3)}(z^{-2m-1}+z^{2m+2}).$$

Then after further simplification and series manipulation we obtain the final result (2.21). We omit these details. This completes the proof of Theorem 2.4.

3. Proof of Dyson's rank conjecture mod 5

3.1. **5-dissections of some theta-products.** Let p be a positive integer and $F(q)$ be a power series in q

$$F(q) = \sum_n a(n)q^n.$$

The *p-dissection* of $F(q)$ splits a series into p parts according to the residue mod p of the exponent of q and is given by

$$F(q) = \sum_{r=0}^{p-1}\sum_{n\equiv r \pmod p} a(n)q^n = \sum_{r=0}^{p-1} q^r F_r(q^p). \tag{3.1}$$

The Atkin U-operators pick out a part of this p-dissection

$$U_{p,r}(F(q)) := F_r(q) = \sum_n a(pn+r)q^n \tag{3.2}$$

for $0 \le r \le p-1$. We also define the operators $U^*_{p,m}$, and $A_{p,m}$ by

$$U^*_{p,m}(F(q)) = \sum_n a(pn+m)q^{pn+m},$$

$$A_{p,m}(F) = \sum_{n} a(n) q^{(n-m)/p},$$

so that

$$U_{p,m} = A_{p,0} \circ U^*_{p,m}.$$

The following 5-dissections are well-known and can be proved with little more than Jacobi's triple product identity (2.3).

Lemma 3.1. *Let* $\zeta = \exp(2\pi i/5)$. *Then*

$$(\zeta q)_\infty (\zeta^{-1} q)_\infty (q)_\infty = J_{25,10} + q(\zeta^2 + \zeta^{-2}) J_{25,5}, \tag{3.3}$$

$$E(q) := (q)_\infty = J_{25} \left(\frac{J_{25,10}}{J_{25,5}} - q - q^2 \frac{J_{25,5}}{J_{25,10}} \right), \tag{3.4}$$

$$\theta_4(q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = J_{50,25} - 2q J_{50,15} + 2q^4 J_{50,5} \tag{3.5}$$

Proof. Equation (3.3) follows from [14, Lemma 3.19]. See [14, Lemma 3.18] for equation (3.4). The general p-dissection of $E(q)$, where $(p, 6) = 1$, is due to Atkin and Swinnerton-Dyer [7, Lemma 6]. The proof depends on Euler's Pentagonal Number Theorem [5, p.11], and Watson's quintuple product identity [7, Lemma 5]. Equation (3.5) follows easily from Jacobi's triple product identity (2.3). ☐

The following results follow easily from Lemma 3.1.

Lemma 3.2.

$$U_{5,2} \left(E(q)^2 \right) = -J_5^2, \tag{3.6}$$

$$U_{5,3} \left(\theta_4(q)\, E(q) \right) = 2 \frac{J_5 J_{5,1} J_{10,3}}{J_{5,2}}, \tag{3.7}$$

$$U_{5,4} \left(\theta_4(q)\, E(q) \right) = 2 \frac{J_5 J_{5,2} J_{10,1}}{J_{5,1}}. \tag{3.8}$$

3.2. **5-dissections involving the rank function and Hecke-Rogers series.** We begin by letting $z = 1$ in (2.18) to obtain the identity

$$(q)_\infty^2 = E(q)^2 = \sum_{n=0}^{\infty} \sum_{j=-\lfloor n/2 \rfloor}^{\lfloor n/2 \rfloor} (-1)^{n+j} q^{\frac{1}{2}(n^2 - 3j^2) + \frac{1}{2}(n-j)}, \tag{3.9}$$

which, as mentioned before, is originally due to L. J. Rogers [22, p.323].

Lemma 3.3. *Let* $\zeta = \exp(2\pi i/5)$. *Then*

$$U_{5,2} \left((\zeta q)_\infty (\zeta^{-1} q)_\infty (q)_\infty \, R(\zeta, q) \right) = -J_5^2.$$

Proof. We have

$$\frac{1}{2}(n^2-3j^2)+\frac{1}{2}(n-j)\equiv 2 \pmod 5$$

if and only if $n\equiv 2 \pmod 5$ and $j\equiv 4 \pmod 5$, in which case $n-3j\equiv 0 \pmod 5$. Similarly

$$\frac{1}{2}(n^2-3j^2)+\frac{1}{2}(n+j)\equiv 2 \pmod 5$$

if and only if $n\equiv 2 \pmod 5$ and $j\equiv 1 \pmod 5$, in which case $n-3j+1\equiv 0 \pmod 5$. Thus from (2.18), (3.9) and (3.6) we have

$$\begin{aligned}
&U_{5,2}\left((\zeta q)_\infty(\zeta^{-1}q)_\infty(q)_\infty\, R(\zeta,q)\right)\\
&= U_{5,2}\left(\sum_{n=0}^{\infty}\sum_{j=-\lfloor n/2\rfloor}^{\lfloor n/2\rfloor}(-1)^{n+j}q^{\frac{1}{2}(n^2-3j^2)+\frac{1}{2}(n-j)}\right)\\
&= U_{5,2}\left(E(q)^2\right) = -J_5^2.
\end{aligned}$$

□

Next we use the Hecke-Rogers identity (2.19). By letting $z=1$ we see that this identity is a z-analog of

$$\theta_4(q)\,E(q)=\frac{(q)_\infty^3}{(q^2;q^2)_\infty}=\sum_{n=0}^{\infty}\sum_{j=-n}^{n}(-1)^j q^{\frac{1}{2}n(3n+1)-j^2}(1-q^{2n+1}). \tag{3.10}$$

Lemma 3.4. *Let* $\zeta=\exp(2\pi i/5)$. *Then*

$$U_{5,3}\left((\zeta q)_\infty(\zeta^{-1}q)_\infty(q)_\infty\, R(\zeta,q^2)\right)=(\zeta^2+\zeta^3)\,\frac{J_5J_{5,1}J_{10,3}}{J_{5,2}}, \tag{3.11}$$

$$U_{5,4}\left((\zeta q)_\infty(\zeta^{-1}q)_\infty(q)_\infty\, R(\zeta,q^2)\right)=(\zeta+\zeta^4)\,\frac{J_5J_{5,2}J_{10,1}}{J_{5,1}}. \tag{3.12}$$

Proof. We have

$$\frac{1}{2}n(3n+1)-j^2\equiv 3 \pmod 5$$

if and only if $n\equiv 1 \pmod 5$ and $j\equiv \pm 2 \pmod 5$, or $n\equiv 2 \pmod 5$ and $j\equiv \pm 2 \pmod 5$. Similarly we have

$$\frac{1}{2}n(3n+1)+(2n+1)-j^2\equiv 3 \pmod 5$$

if and only if $n\equiv 2 \pmod 5$ and $j\equiv \pm 2 \pmod 5$, or $n\equiv 3 \pmod 5$ and $j\equiv \pm 2 \pmod 5$. Thus from (2.19) we have

$$U_{5,3}^{*}\left((\zeta q)_\infty(\zeta^{-1}q)_\infty(q)_\infty\, R(\zeta,q^2)\right)=U_{5,3}^{*}\left(\sum_{n=0}^{\infty}\sum_{j=-n}^{n}(-1)^j\zeta^j q^{\frac{1}{2}n(3n+1)-j^2}(1-q^{2n+1})\right)$$

$$= \zeta^2 \left(\sum_{\substack{n\ge0 \\ n\equiv 1,2 \pmod 5}} \sum_{\substack{-n\le j\le n \\ j\equiv 2 \pmod 5}} (-1)^j q^{\frac12 n(3n+1)-j^2} \right.$$
$$\left. - \sum_{\substack{n\ge0 \\ n\equiv 2,3 \pmod 5}} \sum_{\substack{-n\le j\le n \\ j\equiv 2 \pmod 5}} (-1)^j q^{\frac12 n(3n+1)+2(n+1)-j^2} \right)$$
$$+ \zeta^3 \left(\sum_{\substack{n\ge0 \\ n\equiv 1,2 \pmod 5}} \sum_{\substack{-n\le j\le n \\ j\equiv 3 \pmod 5}} (-1)^j q^{\frac12 n(3n+1)-j^2} \right.$$
$$\left. - \sum_{\substack{n\ge0 \\ n\equiv 2,3 \pmod 5}} \sum_{\substack{-n\le j\le n \\ j\equiv 3 \pmod 5}} (-1)^j q^{\frac12 n(3n+1)+2(n+1)-j^2} \right)$$
$$= \frac12(\zeta^2+\zeta^3) \left(\sum_{\substack{n\ge0 \\ n\equiv 1,2 \pmod 5}} \sum_{\substack{-n\le j\le n \\ j\equiv 2,3 \pmod 5}} (-1)^j q^{\frac12 n(3n+1)-j^2} \right.$$
$$\left. - \sum_{\substack{n\ge0 \\ n\equiv 2,3 \pmod 5}} \sum_{\substack{-n\le j\le n \\ j\equiv 2,3 \pmod 5}} (-1)^j q^{\frac12 n(3n+1)+2(n+1)-j^2} \right)$$
$$= \frac12(\zeta^2+\zeta^3) U_{5,3}^* \left(\sum_{n=0}^{\infty} \sum_{j=-n}^{n} (-1)^j q^{\frac12 n(3n+1)-j^2} (1-q^{2n+1}) \right).$$

Thus by (3.10) and (3.7) we have

$$U_{5,3}\left((\zeta q)_\infty (\zeta^{-1}q)_\infty (q)_\infty \, R(\zeta,q^2) \right) = \frac12(\zeta^2+\zeta^3) U_{5,3}((q)_\infty^3 \, R(1,q^2))$$
$$= \frac12(\zeta^2+\zeta^3) U_{5,3}(\theta_4(q)\, E(q)) = (\zeta^2+\zeta^3)\, \frac{J_5 J_{5,2} J_{10,1}}{J_{5,1}},$$

which is equation (3.11). The proof of (3.12) is analogous. ∎

3.3. **Completing the proof of Dyson's mod 5 rank conjecture.** We start by letting $z = \zeta = \exp(2\pi i/5)$ in the generating function for the rank. We have

$$\begin{aligned} R(\zeta,q) &= \sum_{n=0}^{\infty}\sum_{m} N(m,n)\zeta^m q^n = \sum_{n=0}^{\infty}\sum_{r=0}^{4}\sum_{\substack{m\equiv r \\ \bmod 5}} N(m,n)\zeta^m q^n \\ &= \sum_{n=0}^{\infty}\left(\sum_{r=0}^{4} N(r,5,n)\zeta^r\right) q^n, \end{aligned}$$

and

$$\text{Coefficient of } q^{5n+4} \text{ in} R(\zeta,q) = \sum_{r=0}^{4} N(r,5,5n+4)\zeta^r.$$

Therefore Dyson's mod 5 rank conjecture (1.4) is equivalent to showing

$$U_{5,4}(R(\zeta,q)) = 0. \tag{3.13}$$

See [14, Lemma 2.2]. Since we have identities on both base q and q^2 we instead prove the equivalent identity

$$U_{5,3}(R(\zeta,q^2)) = 0, \tag{3.14}$$

and write the 5-dissection of $R(\zeta,q^2)$:

$$R(\zeta,q^2) = \sum_{k=0}^{4} q^k R_k(q^5).$$

We obtain three linear equations for the functions $R_2(q)$, $R_3(q)$ and $R_4(q)$. In Lemma 3.3 we replace q by q^2 and use (3.3) to find

$$(\zeta^2+\zeta^3)J_{10,2}\, R_2(q) + J_{10,4}\, R_4(q) = -J_{10}^2. \tag{3.15}$$

By equation (3.3) and Lemma 3.4 we have the following two equations.

$$(\zeta^2+\zeta^3)J_{5,1}\, R_2(q) + J_{5,2}\, R_3(q) = (\zeta^2+\zeta^3)\frac{J_{5,1}J_5J_{10,3}}{J_{5,2}}, \tag{3.16}$$

$$(\zeta^2+\zeta^3)J_{5,1}\, R_3(q) + J_{5,2}\, R_4(q) = (\zeta+\zeta^4)\frac{J_{5,2}J_5J_{10,1}}{J_{5,1}}. \tag{3.17}$$

Solving equations (3.15)–(3.17) we find that

$$\begin{aligned} R_3(q) = \frac{1}{D(q)}\big(& J_5\, J_{5,1}^3\, J_{10,2}\, J_{10,3}\, J_{10,4} - 2\, J_5\, J_{5,1}^2\, J_{5,2}\, J_{10,1}\, J_{10,4}^2 + J_5\, J_{5,2}^3\, J_{10,1}\, J_{10,2}\, J_{10,4} \\ & + J_{10}^2\, J_{5,1}^3\, J_{5,2}\, J_{10,4} - J_{10}^2\, J_{5,1}\, J_{5,2}^3\, J_{10,2} - J_{5,2}\,(J_5\, J_{5,1}^2\, J_{10,1}\, J_{10,4}^2 \\ & +J_5\, J_{5,1}\, J_{5,2}\, J_{10,2}^2\, J_{10,3} - J_5\, J_{5,2}^2\, J_{10,1}\, J_{10,2}\, J_{10,4} - J_{10}^2\, J_{5,1}^3\, J_{10,4})\,(\zeta^2+\zeta^3)\big)\,, \end{aligned} \tag{3.18}$$

where

$$D(q) = J_{5,1}^4\, J_{10,4}^2 + J_{5,1}^2\, J_{5,2}^2\, J_{10,2}\, J_{10,4} - J_{5,2}^4\, J_{10,2}^2$$

$$= 1 - 6\,q + 10\,q^2 + 4\,q^3 - 19\,q^4 - 10\,q^6 + 64\,q^7 - 9\,q^8 - 66\,q^9 - 40\,q^{11} + \cdots .$$

We have

$$\frac{J_{10,1}\,J_{10,4}}{J_{10}^2} = \frac{J_{5,1}}{J_5}, \qquad \frac{J_{10,2}\,J_{10,3}}{J_{10}^2} = \frac{J_{5,3}}{J_5}. \tag{3.19}$$

This together with (3.18) and some simplification we have

$$R_3(q) = 0,$$

which completes the proof of (3.14) and thus Dyson's mod 5 rank conjecture (1.4).

It can be shown that $D(q)$ is an eta-quotient

$$D(q) = \frac{J_{10}^3 J_1^6}{J_5^2 J_2} = \frac{1}{q}\,\frac{\eta^3(10\tau)\eta^6(\tau)}{\eta^2(5\tau)\eta(2\tau)}.$$

This can be proved using standard modular function techniques, but this identity is not needed in the proof of (3.14).

Since $R_3(q) = 0$, equations (3.16) and (3.17) imply that $R_2(q)$ and $R_4(q)$ have product forms:

$$R_2(q) = \frac{J_{10}^2}{J_{10,2}}, \quad R_4(q) = -(1 + \zeta^2 + \zeta^3)\,\frac{J_{10}^2}{J_{10,4}}. \tag{3.20}$$

4. Proof of Ramanujan's mod 5 rank identity

Again let $\zeta = \exp(2\pi i/5)$. The following identity appears on p.20 of Ramanujan's Lost Notebook [21].

$$\begin{aligned} R(\zeta, q) = {} & A(q^5) + (\zeta + \zeta^{-1} - 2)\,\phi(q^5) + q\,B(q^5) + (\zeta + \zeta^{-1})\,q^2\,C(q^5) \\ & - (\zeta + \zeta^{-1})\,q^3 \left\{ D(q^5) - (\zeta^2 + \zeta^{-2} - 2)\frac{\psi(q^5)}{q^5} \right\}, \end{aligned} \tag{4.1}$$

where

$$A(q) = \frac{(q^2, q^3, q^5; q^5)_\infty}{(q, q^4; q^5)_\infty^2},\ B(q) = \frac{(q^5; q^5)_\infty}{(q, q^4; q^5)_\infty},\ C(q) = \frac{(q^5; q^5)_\infty}{(q^2, q^3; q^5)_\infty},\ D(q) = \frac{(q, q^4, q^5; q^5)_\infty}{(q^2, q^3; q^5)_\infty^2},$$

and

$$\phi(q) = -1 + \sum_{n=0}^{\infty} \frac{q^{5n^2}}{(q; q^5)_{n+1}(q^4; q^5)_n}, \qquad \psi(q) = -1 + \sum_{n=0}^{\infty} \frac{q^{5n^2}}{(q^2; q^5)_{n+1}(q^3; q^5)_n}.$$

We write the 5-dissection of $R(\zeta, q)$:

$$R(\zeta, q) = \sum_{k=0}^{4} q^k R_k(q^5).$$

From equations (3.13) and (3.20) we have

$$R_1(q) = \frac{J_5^2}{J_{5,1}}, \quad R_2(q) = (\zeta + \zeta^{-1})\,\frac{J_5^2}{J_{5,2}} \quad R_4(q) = 0. \tag{4.2}$$

It suffices to show that

$$R_0(q) = \frac{{J_5}^2 J_{5,2}}{{J_{5,1}}^2} + \left(\zeta^4 + \zeta - 2\right)\,\phi(q), \tag{4.3}$$

$$R_3(q) = -\left(\zeta^4 + \zeta\right)\,\frac{J_{5,1}{J_5}^2}{{J_{5,2}}^2} + \frac{1}{q}\,\left(2\,\zeta^3 + 2\,\zeta^2 + 1\right)\,\psi(q). \tag{4.4}$$

This time we use the Hecke-Rogers identity (2.20). By letting $z = 1$ we see that this is a different z-analog of (3.9). We define

$$\widetilde{R}(z,q) = (1+z)(z^2q;q)_\infty(z^{-2}q;q)_\infty(q;q)_\infty R(z,q). \tag{4.5}$$

Lemma 4.1. *Let $\zeta = \exp(2\pi i/5)$. Then*

$$U_{5,0}\left(\widetilde{R}(\zeta,q)\right) = (1+\zeta)\,\frac{J_5^2\,J_{5,2}^2}{J_{5,1}^2} - (2 + 2\zeta + \zeta^3)\,\left(\widetilde{R}(q,q^5) - J_{5,2}\right), \tag{4.6}$$

$$U_{5,4}\left(\widetilde{R}(\zeta,q)\right) = (\zeta^2 + \zeta^4)\,\frac{J_5^2\,J_{5,1}^2}{J_{5,2}^2} - \frac{1}{q}\,(2 + 2\zeta + \zeta^3)\,\left(\widetilde{R}(q^2,q^5) - J_{5,1}\right). \tag{4.7}$$

Proof. We have

$$\frac{1}{2}(n^2 - 3j^2) + \frac{1}{2}(n-j) \equiv 0 \pmod 5$$

if and only if $(n,j) \equiv (0,0)$, $(0,3)$, $(1,4)$, $(3,4)$, $(4,0)$ or $(4,3)$ (mod 5). We have the following table

$\zeta^{n+1} + \zeta^{-n}$	n	j
$1+\zeta$	0	0
$1+\zeta$	0	3
$-\zeta^3 - \zeta - 1$	1	4
$-\zeta^3 - \zeta - 1$	3	4
$1+\zeta$	4	0
$1+\zeta$	4	3

We let

$$\mathcal{S}_1 = \{(0,0),\,(0,3),\,(1,4),\,(3,4),\,(4,0),\,(4,3)\}, \qquad \mathcal{S}_2 = \{(1,4),\,(3,4)\},$$

so that by (2.20) we have

$$U_{5,0}^{*}\left(\widetilde{R}(\zeta,q)\right) \tag{4.8}$$

$$= (1+\zeta) \sum_{\substack{n=0 \\ (n,j)\in\mathcal{S}_1}}^{\infty} \sum_{\substack{j=-\lfloor n/2\rfloor \\ \pmod 5}}^{\lfloor n/2\rfloor} (-1)^{n+j} q^{\frac{1}{2}(n^2-3j^2)+\frac{1}{2}(n-j)}$$
$$- (2+2\zeta+\zeta^3) \sum_{\substack{n=0 \\ (n,j)\in\mathcal{S}_2}}^{\infty} \sum_{\substack{j=-\lfloor n/2\rfloor \\ \pmod 5}}^{\lfloor n/2\rfloor} (-1)^{n+j} q^{\frac{1}{2}(n^2-3j^2)+\frac{1}{2}(n-j)}$$

Also by (2.20) we have

$$\widetilde{R}(q,q^5) = \sum_{n=0}^{\infty} \sum_{j=-\lfloor n/2\rfloor}^{\lfloor n/2\rfloor} (-1)^{n+j} \left(q^{n+1}+q^{-n}\right) q^{\frac{5}{2}(n^2-3j^2)+\frac{5}{2}(n-j)}. \tag{4.9}$$

Now we let

$$V(n,j) = \frac{1}{2}(n^2-3j^2) + \frac{1}{2}(n-j), \tag{4.10}$$

and find that

$$\begin{aligned} \frac{1}{5}V(5n+1,-5j-1) &= 5\,V(n,j) - n, \\ \frac{1}{5}V(5n+3,-5j-1) &= 5\,V(n,j) + n + 1. \end{aligned} \tag{4.11}$$

We solve some inequalities.

$$-\lfloor \tfrac{1}{2}(5n+1)\rfloor \le -5j-1 \le \lfloor \tfrac{1}{2}(5n+1)\rfloor \iff \begin{cases} -m \le j \le m-1 & \text{if } n=2m, \\ -m \le j \le m & \text{if } n = 2m+1, \end{cases}$$

and

$$-\lfloor \tfrac{1}{2}(5n+3)\rfloor \le -5j-1 \le \lfloor \tfrac{1}{2}(5n+3)\rfloor \iff \begin{cases} -m \le j \le m & \text{if } n=2m, \\ -m-1 \le j \le m & \text{if } n = 2m+1. \end{cases}$$

It follows that

$$\begin{aligned} & A_{5,0}\left(\sum_{\substack{n=0 \\ (n,j)\in\mathcal{S}_2}}^{\infty} \sum_{\substack{j=-\lfloor n/2\rfloor \\ \pmod 5}}^{\lfloor n/2\rfloor} (-1)^{n+j} q^{\frac{1}{2}(n^2-3j^2)+\frac{1}{2}(n-j)} \right) \\ &= \sum_{n=0}^{\infty} \sum_{j=-\lfloor n/2\rfloor}^{\lfloor n/2\rfloor} (-1)^{n+j} \left(q^{n+1}+q^{-n}\right) q^{\frac{5}{2}(n^2-3j^2)+\frac{5}{2}(n-j)} - \sum_{m=-\infty}^{\infty} (-1)^m q^{m(5m+1)/2} \\ &= \widetilde{R}(q,q^5) - J_{5,2}, \end{aligned} \tag{4.12}$$

by (2.20), (4.5) and Jacobi's triple product identity (2.3). Now

$$\sum_{\substack{n=0 \\ (n,j)\in\mathcal{S}_1}}^{\infty} \sum_{\substack{j=-\lfloor n/2\rfloor \\ \pmod 5}}^{\lfloor n/2\rfloor} (-1)^{n+j} q^{\frac{1}{2}(n^2-3j^2)+\frac{1}{2}(n-j)} \tag{4.13}$$
$$= U^*_{5,0}\left(\sum_{n=0}^{\infty}\sum_{j=-\lfloor n/2\rfloor}^{\lfloor n/2\rfloor} (-1)^{n+j} q^{\frac{1}{2}(n^2-3j^2)+\frac{1}{2}(n-j)}\right)$$
$$= U^*_{5,0}\left((q)^2_\infty\right) \qquad \text{(by (3.9))}$$
$$= \frac{J^2_{25}\,J^2_{25,10}}{J^2_{25,5}},$$

by (3.4). Thus from (4.8), (4.12) and (4.13) we have

$$U_{5,0}\left(\widetilde{R}(\zeta,q)\right) = (1+\zeta)\,\frac{J^2_5\,J^2_{5,2}}{J^2_{5,1}} - (2+2\zeta+\zeta^3)\,\left(\widetilde{R}(q,q^5) - J_{5,2}\right), \tag{4.14}$$

which is (4.6). The proof of (4.7) is analogous. □

Now by (4.5) and the definition of $\phi(q)$ we have

$$\widetilde{R}(q,q^5) = (q^2;q^5)_\infty(q^3;q^5)_\infty(q^5;q^5)_\infty \frac{1}{1-q} R(q,q^5) \tag{4.15}$$
$$= J_{5,2}\sum_{n=0}^{\infty} \frac{q^{5n^2}}{(q;q^5)_{n+1}(q^4;q^5)_n} \qquad \text{(by Jacobi's triple product (2.3))}$$
$$= J_{5,2}\,\left(1+\phi(q)\right).$$

By replacing z by ζ in (4.5), ζ by ζ^2 in (3.3) and by (4.2) we have

$$\widetilde{R}(\zeta,q) \tag{4.16}$$
$$= (1+\zeta)\left(J_{25,10} + q(\zeta+\zeta^4)J_{25,5}\right)\left(R_0(q^5) + q\frac{J^2_{25}}{J_{25,5}} + q^2(\zeta+\zeta^4)\frac{J^2_{25}}{J_{25,10}} + q^3R_3(q^5)\right)$$

We apply $U_{5,0}$ to both sides of (4.16) to find that

$$U_{5,0}\left(\widetilde{R}(\zeta,q)\right) = (1+\zeta)\,J_{5,2}\,R_0(q). \tag{4.17}$$

By (4.17) and (4.14) we have

$$(1+\zeta)\,J_{5,2}\,R_0(q) = (1+\zeta)\frac{J^2_5 J^2_{5,2}}{J^2_{5,1}} - (2+\zeta+\zeta^3)\,J_{5,2}\,\phi(q),$$

and we easily deduce (4.3). The proof of (4.4) is similar.

5. Equations for the rank mod 7

In this section we consider Dyson's mod 7 rank conjecture (1.5). The following identity is the corresponding analog of (3.3). Through this section we assume $\zeta = \exp(2\pi i/7)$.

$$(\zeta q)_\infty(\zeta^{-1}q)_\infty(q)_\infty = J_{49,21} + q(\zeta^2+\zeta^3+\zeta^4+\zeta^5)J_{49,14} - q^3(\zeta^3+\zeta^4)J_{49,7}. \tag{5.1}$$

The proof of the following lemma is analogous to the proof of Lemma 3.4 and depends on the Hecke-Rogers identities (2.18), (2.20) and (2.21).

Lemma 5.1. *Let $\zeta = \exp(2\pi i/7)$. Then*

$$U_{7,4}\left((\zeta q)_\infty(\zeta^{-1}q)_\infty(q)_\infty\, R(\zeta,q)\right) = J_7^2,$$
$$U_{7,4}\left((1+\zeta)(\zeta^2 q)_\infty(\zeta^{-2}q)_\infty(q)_\infty\, R(\zeta,q)\right) = 2\zeta^4\, J_7^2$$
$$U_{7,3}\left((1+\zeta)(\zeta^2q^2;q^2)_\infty(\zeta^{-2}q^2;q^2)_\infty(q^2;q^2)_\infty\, R(\zeta,q)\right) = 2\zeta^4\,\frac{J_{14}^3}{J_7}.$$

We write the 7-dissection of $R(\zeta,q)$:

$$R(\zeta,q) = \sum_{k=0}^{6} q^k R_k(q^7).$$

Dyson's mod 7 rank conjecture is equivalent to showing that

$$U_{7,5}\left(R(\zeta,q)\right) = R_5(q) = 0.$$

Unfortunately we have only been able to find three linear equations for the functions $R_1(q)$, $R_3(q)$ and $R_4(q)$.

$$-\,(\zeta^4+\zeta^3)\,J_{7,1}\,R_1(q) + (\zeta^5+\zeta^4+\zeta^3+\zeta^2)\,J_{7,2}\,R_3(q) + J_{7,3}\,R_4(q) = J_7^2, \tag{5.2}$$

$$(\zeta^5+\zeta^4+\zeta^3)\,J_{7,1}\,R_1(q) + \zeta^4\,J_{7,2}\,R_3(q) + (\zeta+1)\,J_{7,3}\,R_4(q) = 2\,\zeta^4\,J_7^2, \tag{5.3}$$

$$\zeta^4\,J_{14,4}\,R_1(q) + (\zeta+1)\,J_{14,6}\,R_3(q) + (\zeta^5+\zeta^4+\zeta^3)\,q\,J_{14,2}\,R_4(q) = 2\,\zeta^4\,q\,\frac{J_{14}^3}{J_7}. \tag{5.4}$$

Using these equations it is possible to show that

$$R_1(q) = \frac{J_7^2}{J_{7,1}},\quad R_3(q) = (\zeta^5+\zeta^2+1)\,\frac{J_7^2}{J_{7,2}},\quad R_4(q) = -(\zeta^5+\zeta^2)\,\frac{J_7^2}{J_{7,3}}.$$

We have been unable to find a fourth linear equation only involving $R_1(q)$, $R_3(q)$, $R_4(q)$, $R_5(q)$. This should be compared with equations (3.15), (3.16) and (3.17), which were enough to prove the mod 5 conjecture (1.4).

6. Conclusion

In this paper we presented a new approach to proving Dyson's rank conjectures. This new approach involved utilising various Hecke-Rogers identities. We showed how this method gave a new proof for Dyson's mod 5 rank conjecture as well as the related identity in Ramanujan's Lost Notebook. We end by listing some problems.

1. Extend the methods of this paper to prove Dyson's mod 7 rank conjecture (1.5), and find a new proof for the mod 7 analog of Ramanujan's identity (4.1). See [4, p.16].
2. Find simple proofs of the four Hecke-Rogers identities (2.18)–(2.21). Combinatorial proofs are also needed. These results would lead to a truly elementary proof of Dyson's rank conjectures.
3. Apply the methods of this paper to other rank-type functions, including Lovejoy's overpartition rank [20], Berkovich and the author's M_2-rank [8], and Jenning-Shaffer's exotic Bailey-Slater spt-functions [18].

Data Availability Statement. Data sharing not applicable to this article as the research of this paper does not involve the use of any datasets.

Acknowledgments. The author would like to thank Jonathan Bradley-Thrush for showing how his Theorem 2.1 can be used to prove the Hecke-Rogers identities (2.19) and (2.21).

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