CONGRUENCES MODULO 4 FOR WEIGHT 3/2 ETA-PRODUCTS

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ABSTRACT. We find and prove a class of congruences modulo 4 for eta-products associated with certain ternary quadratic forms. This study was inspired by similar conjectured congruences modulo 4 for certain mock theta functions.

1. INTRODUCTION

Let $p(n)$ be the number of unrestricted partitions of n. Ramanujan discovered and later proved that

$$p(5n+4) \equiv 0 \pmod{5}, \quad p(7n+5) \equiv 0 \pmod{7}, \quad \text{and} \quad p(11n+6) \equiv 0 \pmod{11}. \tag{1.1}$$

Congruences of this type are called Ramanujan-type congruences. Many authors have considered Ramanujan-type congruences for the coefficients of modular forms and more recently the coefficients of mock theta functions. For a given prime ℓ there are standard techniques for proving such congruences. A more difficult problem is proving Ramanujan-type congruences for a given partition-type function for infinitely many arithmetic progressions. This paper arose after a study of some conjectures mod 4 for the coefficients of certain mock theta functions. Define sequences $u(n)$, $v(n)$ by

$$\sum_{n=0}^{\infty} u(n)q^n = \sum_{n=0}^{\infty} (-q;q)_n^2 q^{n+1}, \tag{1.2}$$

$$\sum_{n=0}^{\infty} v(n)q^n = \sum_{n=0}^{\infty} \frac{(-q;q)_n^2 q^n}{(q;q^2)_{n+1}}, \tag{1.3}$$

where as usual

$$(a)_n = (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k).$$

We have the following conjectures.

2010 *Mathematics Subject Classification.* 11E20, 11F33.
Key words and phrases. ternary quadratic forms, Ramanujan-type congruences.
The second author was supported in part by a grant from the Simon's Foundation (#318714).

Conjecture 1.1 (Bryson, Ono, Pitman and Rhoades [5]). *If $p \equiv 7, 11, 13, 17 \pmod{24}$ is a prime and $\left(\frac{k}{p}\right) = -1$, then for all n we have*

$$u(p^2n + kp - \delta(p)) \equiv 0 \pmod{4}, \tag{1.4}$$

where $\delta(p) = (p^2 - 1)/24$.

Conjecture 1.2 (Kim, Lim and Lovejoy [12]). *Let $p \not\equiv -1 \pmod{8}$ be prime and let k be a positive integer satisfying $p \parallel 8k + 7$. If $(8k + 7)/p$ is a quadratic residue modulo p, then*

$$v(p^2n + k) \equiv 0 \pmod{4}.$$

In order to understand these conjectures we led to determine whether similar congruences occur naturally in the theory of modular forms. We prove congruences mod 4 for the coefficients of three certain eta-products associated with ternary quadratic forms. We define

$a(n)$ = the number of representations of n as a sum of two pentagonal numbers and three times a triangular number,

$b(n)$ = the number of representations of n as a sum of a pentagonal number and three times the sum of two triangular numbers,

$c(n)$ = the number of representations of n as a sum of a pentagonal number and two triangular numbers,

so that

(1.5)

$$f(q) := \sum_{n=0}^{\infty} a(n)q^n = \left(\sum_{k=-\infty}^{\infty} q^{k(3k+1)/2}\right)^2 \sum_{m=0}^{\infty} q^{3m(m+1)/2} = \frac{J_3^3 J_2^2}{J_1^2} = q^{-11/24}\frac{\eta(3\tau)^3\eta(2\tau)^2}{\eta(\tau)^2},$$

$$\sum_{n=0}^{\infty} b(n)q^n = \sum_{k=-\infty}^{\infty} q^{k(3k+1)/2} \left(\sum_{m=0}^{\infty} q^{3m(m+1)/2}\right)^2 = \frac{J_6^3 J_2}{J_1} = q^{-19/24}\frac{\eta(6\tau)^3\eta(2\tau)}{\eta(\tau)},$$

$$\sum_{n=0}^{\infty} c(n)q^n = \sum_{k=-\infty}^{\infty} q^{k(3k+1)/2} \left(\sum_{m=0}^{\infty} q^{m(m+1)/2}\right)^2 = \frac{J_3^2 J_2^5}{J_6 J_1^3} = q^{-7/24}\frac{\eta(3\tau)^2\eta(2\tau)^5}{\eta(6\tau)\eta(\tau)^3}.$$

Here we have used the usual notation for infinite products and the Dedekind eta-function

$$J_k = \prod_{n=1}^{\infty}(1 - q^{kn}), \qquad \eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1 - q^n),$$

where $q = \exp(2\pi i\tau)$ and $\Im(\tau) > 0$. Also we note the well-known identities

$$\sum_{k=-\infty}^{\infty} q^{k(3k+1)/2} = \frac{J_3^2 J_2}{J_6 J_1}, \qquad \sum_{k=0}^{\infty} q^{k(k+1)/2} = \frac{J_2^2}{J_1},$$

which follow from Jacobi's triple product identity [2, p. 35]

$$\sum_{n=-\infty}^{\infty} z^n q^{n(n-1)/2} = \prod_{n=1}^{\infty}(1-zq^{n-1})(1-z^{-1}q^n)(1-q^n).$$

We will prove the following congruences which is the main result in this paper.

Theorem 1.3. *Let $p>3$ be prime, suppose $24\delta_p \equiv 1 \pmod{p^2}$, and k, $n\in\mathbb{Z}$ where $\left(\frac{k}{p}\right)=1$. Then*

(i) $$a(p^2n+(pk-11)\delta_p) \equiv 0 \pmod 4, \qquad \text{if } p \not\equiv 11 \pmod{24},$$

(ii) $$b(p^2n+(pk-19)\delta_p) \equiv 0 \pmod 4, \qquad \text{if } p \not\equiv 19 \pmod{24},$$

(iii) $$c(p^2n+(pk-7)\delta_p) \equiv 0 \pmod 4, \qquad \text{if } p \not\equiv 7 \pmod{24}.$$

Examples. We illustrate (i) when $p=5$, 7. We have

$$a(25n+6)\equiv 0 \pmod 4, \qquad a(25n+16)\equiv 0 \pmod 4,$$

and

$$a(49n+8)\equiv 0 \pmod 4, \qquad a(49n+15)\equiv 0 \pmod 4, \qquad a(49n+43)\equiv 0 \pmod 4.$$

We relate $a(n)$ to $r_3(n)$, the number of representations of n as a sum of three squares. We have

$$\sum_{n=0}^{\infty} r_3(n)q^n = \sum_{x,y,z\in\mathbb{Z}} q^{x^2+y^2+z^2}. \tag{1.6}$$

Noting that

$$\sum_{m=0}^{\infty} q^{3m(m+1)/2} = \sum_{m=-\infty}^{\infty} q^{6m^2-3m},$$

we let

$$F(q) := \sum_{n=0}^{\infty} A(n)q^n := q^{11} f(q^{24}) = \sum_{x,y,z\in\mathbb{Z}} q^{(6x+1)^2+(6y+1)^2+9(4z+1)^2}, \tag{1.7}$$

so that

$$a(n) = A(24n+11), \tag{1.8}$$

for $n\ge 0$. Clearly $A(n)=0$ if $n\not\equiv 11 \pmod{24}$. We observe that

$$x^2+y^2+z^2 \equiv 11 \pmod{24}$$

if and only if one of x^2, y^2, z^2 congruent to 9 and the others congruent to 1 modulo 24. Since $x^2\equiv 1 \pmod{24}$ if and only if $x\equiv \pm 1 \pmod 6$ and also $x^2\equiv 9 \pmod{24}$ if and only if $x\equiv \pm 3 \pmod 6$,

$$\sum_{n=0}^{\infty} r_3(24n+11)q^{24n+11} = 3\sum_{u,v=\pm 1}\sum_{x,y,z\in\mathbb{Z}} q^{(6x+u)^2+(6y+v)^2+9(2z+1)^2}$$

$$=24\sum_{x,y,z\in\mathbb{Z}} q^{(6x+1)^2+(6y+1)^2+9(4z+1)^2},$$

Therefore

$$24A(n)=\begin{cases} r_3(n) & \text{if } n\equiv 11 \pmod{24},\\ 0 & \text{otherwise.}\end{cases} \tag{1.9}$$

In Sections 3 and 4 we prove case (i) of Theorem 1.3 in detail. Section 3 deals with $A(n)$ when n is square-free. In Section 4 we study $A(n)$ when is not square free. We sketch the proof of the remaining cases (ii), (iii) in Section 5.

2. Preliminary results for $r_3(n)$ and the class number

The number of representations of n as a sum of three squares, ternary quadratic forms and the relation to the class number dates back to work of Gauss [8]. We need results of Gauss for $r_3(n)$ when $n \equiv 3 \pmod 8$.

Theorem 2.1 ([10, p.51])**.** *If n is square-free, $n > 3$ and $n \equiv 3 \pmod 8$, then we have*

$$r_3(n) = 24h(-n),$$

where $h(-n)$ is the class number of $\mathbb{Q}(\sqrt{-n})$.

This theorem implies

Theorem 2.2 ([10, p.52])**.** *If n is square-free, $n > 3$ and $n \equiv 3 \pmod 8$, then we have*

$$r_3(n) = 2^{t+2}k,$$

where t is the number of distinct prime factors of n and k is the number of classes in each genus of $\mathbb{Q}(\sqrt{-n})$.

The following theorem is a consequence of Dirichlet's class number formula [4, p. 346] and Theorem 2.2.

Theorem 2.3 ([10, p.53])**.** *If n is square-free, $n > 3$ and $n \equiv 3 \pmod 8$, then we have*

$$r_3(n) = 8\sum_{r=1}^{(n-1)/2}\left(\frac{r}{n}\right).$$

Here and throughout this paper $\left(\frac{\cdot}{\cdot}\right)$ denotes the Kronecker symbol.

3. Congruences of $A(n)$ for square-free numbers n

Throughout this section (unless otherwise stated) we assume that $n \equiv 11 \pmod{24}$ and is square-free with prime factorization

$$n = \prod_{i=1}^{s} p_i. \tag{3.1}$$

Here $s = s(n)$ is the number of distinct prime divisors of n. From Theorem 2.3 and (1.9) we have

$$3A(n) = \sum_{r=1}^{(n-1)/2} \left(\frac{r}{n}\right).$$

If n is prime we have

$$\sum_{r=1}^{(n-1)/2} \left(\frac{r}{n}\right) \equiv 1 \pmod{2}.$$

This implies

Lemma 3.1. *If $n \equiv 11 \pmod{24}$ is prime then*

$$A(n) \equiv 1 \pmod{2}.$$

If $s \geq 3$, then $2^5 \mid r(n)$ by Theorem 2.2. Hence we have

Lemma 3.2. *If $n \equiv 11 \pmod{24}$ has a least $s = 3$ distinct prime divisors then*

$$A(n) \equiv 0 \pmod{4}.$$

The interesting case is $s = 2$, i.e. $n = p_1 p_2$ is a product of two distinct primes. Since $n \equiv 3 \pmod{4}$, one of p_1 and p_2 is congruent to 1 and the other is congruent to 3 modulo 4. By Gauss's Law of Quadratic Reciprocity we have

$$\left(\frac{p_1}{p_2}\right) = \left(\frac{p_2}{p_1}\right).$$

For the case $s = 2$ we need some additional lemmas. We define

$$D := D(p_1, p_2) := \left\{1 \leq r \leq (p_1 p_2 - 1)/2 : (r, p_1 p_2) = 1\right\}. \tag{3.2}$$

We have

$$3A(p_1 p_2) = \sum_{r=1}^{(n-1)/2} \left(\frac{r}{p_1}\right)\left(\frac{r}{p_2}\right) = \sum_{r \in D} \left(\frac{r}{p_1}\right)\left(\frac{r}{p_2}\right). \tag{3.3}$$

The number of elements in D is

$$N := N(p_1, p_2) := |D| = \frac{1}{2}\varphi(n) = \frac{1}{2}(p_1 - 1)(p_2 - 1),$$

where $\varphi(n)$ is the Euler function. Since one of p_1 and p_2 is congruent to 1 modulo 4, we have

$$4 | N. \tag{3.4}$$

Lemma 3.3. *Let p_1, p_2 be distinct primes with $p_1p_2 \equiv 3 \pmod 4$. If $\left(\frac{p_1}{p_2}\right) = 1$ then*

$$\sum_{r\in D}\left(\frac{r}{p_1}\right) = \sum_{r\in D}\left(\frac{r}{p_2}\right) = 0.$$

Proof. Let

$$E := E(p_1,p_2) := \{r = u + vp_2 : 1 \le u \le p_2 - 1, 0 \le v \le (p_1-3)/2\}.$$

It's easy to check that

$$D \cup D_0 = E \cup E_0 = H, \tag{3.5}$$

$$D \cap D_0 = E \cap E_0 = \emptyset, \tag{3.6}$$

where

$$D_0 := D_0(p_1,p_2) := \left\{up_1 : 1 \le u \le \frac{1}{2}(p_2-1)\right\},$$

$$E_0 := E_0(p_1,p_2) := \left\{u + \frac{1}{2}(p_1-1)p_2 : 1 \le u \le \frac{1}{2}(p_2-1)\right\},$$

$$H := H(p_1,p_2) := \{1 \le r \le (p_1p_2-1)/2 : (r,p_2) = 1\}.$$

From the definition of set E

$$\sum_{r\in E}\left(\frac{r}{p_2}\right) = 0. \tag{3.7}$$

By (3.5) and (3.6) we have

$$\sum_{r\in E}\left(\frac{r}{p_2}\right) + \sum_{r\in E_0}\left(\frac{r}{p_2}\right) = \sum_{r\in D}\left(\frac{r}{p_2}\right) + \sum_{r\in D_0}\left(\frac{r}{p_2}\right). \tag{3.8}$$

We find that

$$\sum_{r\in D_0}\left(\frac{r}{p_2}\right) = \sum_{u=1}^{(p_2-1)/2}\left(\frac{up_1}{p_2}\right) = \left(\frac{p_1}{p_2}\right)\sum_{u=1}^{(p_2-1)/2}\left(\frac{u}{p_2}\right) = \left(\frac{p_1}{p_2}\right)\sum_{r\in E_0}\left(\frac{r}{p_2}\right). \tag{3.9}$$

Since $\left(\frac{p_1}{p_2}\right) = 1$, (3.7)–(3.9) imply that

$$\sum_{r\in D}\left(\frac{r}{p_2}\right) = 0.$$

Similarly we have

$$\sum_{r\in D}\left(\frac{r}{p_1}\right) = 0.$$

$\square$

Lemma 3.4. *With above p_1, p_2 and D, if $(\frac{p_1}{p_2}) = -1$ with $p_1 \equiv 1 \pmod 4$ and $p_2 \equiv 3 \pmod 4$, then*

$$\sum_{r\in D} \left(\frac{r}{p_1}\right) \equiv 0 \pmod 4 \quad \textit{and} \quad \sum_{r\in D} \left(\frac{r}{p_2}\right) \equiv 2 \pmod 4.$$

Proof. We proceed as in the proof of Lemma 3.3 defining E, E_0, and D_0 as before. By (3.7)–(3.9) we have

$$\sum_{r\in D} \left(\frac{r}{p_2}\right) = 2 \sum_{r\in E_0} \left(\frac{r}{p_2}\right),$$

since $(\frac{p_1}{p_2}) = -1$. From the definition of E_0

$$\sum_{r\in E_0} \left(\frac{r}{p_2}\right) \equiv |E_0(p_1, p_2)| = \frac{1}{2}(p_2 - 1) \equiv 1 \pmod 2.$$

Hence

$$\sum_{r\in D} \left(\frac{r}{p_2}\right) = 2 \sum_{r\in E_0} \left(\frac{r}{p_2}\right) \equiv p_2 - 1 \equiv 2 \pmod 4.$$

Now we let $E_0' = E_0(p_2, p_1)$. Then since $(\frac{p_2}{p_1}) = (\frac{p_1}{p_2}) = -1$ we have as before

$$\sum_{r\in D} \left(\frac{r}{p_1}\right) = 2 \sum_{r\in E_0'} \left(\frac{r}{p_1}\right).$$

But

$$\sum_{r\in E_0} \left(\frac{r}{p_1}\right) \equiv |E_0'| = \frac{1}{2}(p_1 - 1) \equiv 0 \pmod 2.$$

Therefore

$$\sum_{r\in D} \left(\frac{r}{p_1}\right) \equiv 0 \pmod 4.$$

□

The following lemma is the main result in this section.

Lemma 3.5. *Suppose $n \equiv 11 \pmod{24}$ is the product of two primes $n = p_1p_2$. We have the following.*

(i) $$A(p_1p_2) \equiv 0 \pmod 4, \quad \textit{if} \left(\frac{p_1}{p_2}\right) = 1,$$

(ii) $$A(p_1p_2) \equiv 2 \pmod 4, \quad \textit{if} \left(\frac{p_1}{p_2}\right) = -1.$$

Proof. **(i).** We assume $\left(\frac{p_1}{p_2}\right) = 1$. For $p \in \{p_1, p_2\}$ and $\varepsilon \in \{+, -\}$ we define $N^{\varepsilon}(p)$ to the number of $r \in D$ with $\left(\frac{r}{p}\right) = \varepsilon$. Here $D = D(p_1, p_2)$ is defined in (3.2). By Lemma 3.3 we have

$$N^+(p_1) = N^-(p_1) = N^+(p_2) = N^-(p_2) = \frac{1}{2}N,$$

where $N = |D|$. For $\varepsilon_1, \varepsilon_2 \in \{+, -\}$ we define

$$M(p_1^{\varepsilon_1}, p_2^{\varepsilon_2}) := \left\{ r \in D \, : \, \left(\frac{r}{p_1}\right) = \varepsilon_1, \quad \left(\frac{r}{p_2}\right) = \varepsilon_2 \right\}.$$

Letting $M(p_1^+, p_2^+) = k$, we have

$$\begin{aligned}
M(p_1^+, p_2^-) &= N^+(p_1) - k = \frac{1}{2}N - k,\\
M(p_1^-, p_2^+) &= N^+(p_2) - k = \frac{1}{2}N - k,\\
M(p_1^-, p_2^-) &= N - M(p_1^+, p_2^+) - M(p_1^+, p_2^-) - M(p_1^-, p_2^+) = k.
\end{aligned}$$

Hence

$$\sum_{r \in D} \left(\frac{r}{p_1}\right)\left(\frac{r}{p_2}\right) = M(p_1^+, p_2^+) + M(p_1^-, p_2^-) - M(p_1^-, p_2^+) - M(p_1^+, p_2^-) = 4k - N.$$

By (3.3) and (3.4) we have

$$3A(n) \equiv 4k - N \equiv 0 \pmod{4},$$

and this completes the proof of (i).

(ii). We assume $p_1 \equiv 1 \pmod{4}$, $p_2 \equiv 3 \pmod{4}$ and $\left(\frac{p_1}{p_2}\right) = -1$. By Lemma 3.4 we have

$$N^+(p_1) \equiv N^-(p_1) \pmod{4},$$

so that

$$2N^+(p_1) \equiv N^+(p_1) + N^-(p_1) = N \equiv 0 \pmod{4}.$$

Also

$$N^+(p_2) \equiv N^-(p_2) + 2 \pmod{4},$$

so that

$$2N^+(p_2) \equiv N^+(p_1) + N^-(p_2) + 2 = N + 2 \equiv 2 \pmod{4}.$$

Letting $M(p_1^+, p_2^+) = k$, we have

$$\begin{aligned}
M(p_1^+, p_2^-) &= N^+(p_1) - k,\\
M(p_1^-, p_2^+) &= N^+(p_2) - k,\\
M(p_1^-, p_2^-) &= N - N^+(p_1) - N^+(p_2) + k.
\end{aligned}$$

Hence

$$\begin{aligned}\sum_{r\in D}\left(\frac{r}{p_1}\right)\left(\frac{r}{p_2}\right) &= M(p_1^+,p_2^+)+M(p_1^-,p_2^-)-M(p_1^-,p_2^+)-M(p_1^+,p_2^-)\\ &= 4k+N-2N^+(p_1)-2N^+(p_2)\\ &\equiv 2 \pmod 4.\end{aligned}$$

□

We combine Lemmas 3.1, 3.2 and 3.5 into

Theorem 3.6. *Suppose $n \equiv 11 \pmod{24}$ is square-free and $A(n)$ is defined by (1.7). We have*

(i) *$A(n)$ is odd if and only if n is a prime,*
(ii) *$A(n) \equiv 2 \pmod 4$ if and only if $n = p_1p_2$ is a product of two primes which satisfy*

$$\left(\frac{p_1}{p_2}\right) = -1.$$

4. General congruences for $A(n)$

In the previous section we considered the congruences for $A(n)$ modulo 2 and 4 assuming n is square-free. In this section we remove this restriction. we use the following identity to complete the congruence of $A(n)$. By [11, p. 101], (1.6) and (1.9), we have

$$A(p^2n)+\left(\frac{-n}{p}\right)A(n)+pA(n/p^2)=(p+1)A(n), \tag{4.1}$$

for $n \equiv 11 \pmod{24}$ and any prime $p > 3$.

We use (4.1) to prove the following two lemmas.

Lemma 4.1. *Suppose $n \equiv 11 \pmod{24}$ and p is any prime satisfying $(p, 6n) = 1$. Then*

(i) *$A(p^2n)$ is odd if and only if $A(n)$ is odd,*
(ii) *$A(p^3n)$ is always even.*

Proof. **(i)** Equation (4.1) implies

$$A(p^2n) \equiv A(n) \pmod 2,$$

since $(n, p) = 1$ and p is odd.

(ii). Replacing n by pn in (4.1) we have

$$A(p^3n) \equiv 0 \pmod 2,$$

again since $(n, p) = 1$ and p is odd. □

Lemma 4.2. *Suppose $p > 3$ is prime. Then $A(p^4n)$ is odd if and only if $A(n)$ is odd.*

Proof. Replacing n by p^2n in (4.1) we have

$$A(p^4n) \equiv A(n) \pmod 2,$$

since p is odd. □

Now we can extend Theorem 3.6i to general n.

Theorem 4.3. *$A(n)$ is odd if and only if n has the form*

$$n = p^{4a+1}m^2,$$

where $p \equiv 11 \pmod{24}$ is prime, and m and a are integers satisfying $(m, 6p) = 1$ and $a \geq 0$.

Proof. ($\Leftarrow$) Suppose $p \equiv 11 \pmod{24}$ is prime, and m and a are integers satisfying $(m, 6p) = 1$ and $a \geq 0$. By Theorem 3.6 and Lemmas 4.1 and 4.2 we have

$$A(p^{4a+1}m^2) \equiv A(pm^2) \equiv A(p) \equiv 1 \pmod 2.$$

($\Rightarrow$) Assume that $A(n)$ is odd. Then $n \equiv 11 \pmod{24}$ has the prime factorization

$$n = \prod_{i=1}^{s} p_i^{\alpha_i}.$$

By Lemma 4.2 we have

$$A(n) \equiv A(n_1) \pmod 2,$$

where $n = n_1 t^4$ for some integer t such that

$$n_1 = \prod_{i=1}^{s} p_i^{\beta_i},$$

with each $0 \leq \beta_i \leq 3$. By Lemma 4.1 $A(n_1)$ is even is $\beta_i = 3$ for some i. Hence $0 \leq \beta_i \leq 2$ for each i. Now let

$$n_2 = \prod_{\beta_i=1} p_i.$$

Then by Lemma 4.1 we have

$$A(n_1) \equiv A(n_2) \pmod 2,$$

which implies that $A(n_2)$ is odd. Theorem 3.6 implies that n_2 is a prime so that

$$n = n_2 t^4 \prod_{\beta_i=2} p_i^2.$$

Noting that $(n_2, p_i) = 1$ for $\beta_i = 2$, we can see that n has the form $n = p^{4a+1}m^2$. □

Finally we extend Theorem 3.6ii to general n. We need some properties of $A(n)$ modulo 4 from (4.1). We omit the proof of the following two lemmas and theorem since their proof is similar to that of Lemmas 4.1 and 4.2 and Theorem 4.3.

Lemma 4.4. *Suppose $n \equiv 11 \pmod{24}$, p is any prime satisfying $(p, 6n) = 1$, and $A(n)$ is even. Then*

(i) $A(p^2n) \equiv A(n) \pmod 4$,
(ii) $A(p^3n) \equiv 0 \pmod 4$.

Lemma 4.5. *Suppose $n \equiv 11 \pmod{24}$, p is any prime satisfying $(p,6)=1$, and $A(n)$ is even. Then*

$$A(p^4n) \equiv A(n) \pmod 4.$$

Theorem 4.6. $A(n) \equiv 2 \pmod 4$ *if and only if n has the form*

$$n = p_1^{4a+1} p_2^{4b+1} m^2,$$

where p_1 and p_2 are primes such that $\left(\frac{p_1}{p_2}\right) = -1$, $p_1p_2 \equiv 11 \pmod{24}$, $(m, 6p_1p_2) = 1$ and $a, b \geq 0$ are integers.

5. Proof of Theorem 1.3(i)

In this section we prove our main theorem on congruences for $a(n)$, which is given by (1.5). We recast Theorem 1.3(i) in terms of $A(n)$, which is defined in (1.7) and related to $a(n)$ by (1.8).

Theorem 5.1. *Suppose $p > 3$ is prime and $p \not\equiv 11 \pmod{24}$. Then for $n \geq 0$ we have*

$$A(p^2n + pk) \equiv 0 \pmod 4,$$

provided

$$\left(\frac{k}{p}\right) = 1.$$

Proof. Suppose $p > 3$ is prime, $p \not\equiv 11 \pmod{24}$, $n \geq 0$ and $\left(\frac{k}{p}\right) = 1$. We let $N = p^2n + pk = p(pn+k)$ so that $p \parallel N$. By Theorem 4.3, $A(N)$ is even. Now suppose $A(N) \equiv 2 \pmod 4$. Then by Theorem 4.6,

$$pn + k = p_2^{4b+1} m^2,$$

for some nonnegative integers b and m and some prime p_2 satisfying $\left(\frac{p}{p_2}\right) = -1$. But

$$\left(\frac{p_2}{p}\right) = \left(\frac{p_2 m^2}{p}\right) = \left(\frac{pn+k}{p}\right) = \left(\frac{k}{p}\right) = 1,$$

which is a contradiction. Hence

$$A(N) = A(p^2n + pk) \equiv 0 \pmod 4.$$

□

We show how Theorem 1.3(i) follows easily from Theorem 5.1. Suppose $p > 3$ be prime, $24\delta_p \equiv 1 \pmod{p^2}$, and k, $n \in \mathbb{Z}$ where $\left(\frac{k}{p}\right) = 1$. By (1.8) we have

$$\begin{aligned} a(p^2n + (pk-11)\delta_p) &= A(24(p^2n + (pk-11)\delta_p) + 11) \\ &= A\left(p^2\left(24n + \tfrac{11}{p^2}(24\delta_p - 1)\right) + pk\right) \equiv 0 \pmod 4, \end{aligned}$$

by Theorem 5.1, since $p \not\equiv 11 \pmod{24}$. We note that $\frac{11}{p^2}(24\delta_p - 1) \in \mathbb{Z}$ since $24\delta_p \equiv 1 \pmod{p^2}$.

6. Congruences for other functions and concluding remarks

In this section we consider other weight 3/2 eta-products with non-trivial congruences modulo 4. Theorem 1.3 contains three eta-products. The proof of Theorem 1.3(i) was completed in Section 5 after preparations in Sections 2, 3, and 4. We omit the proof of Theorem 1.3(ii) as it is completely analogous to that of part (i).

We sketch some details of the proof of part (iii) of Theorem 1.3(iii). This time $r_3(24n+7) = 0$ so we instead consider a different ternary quadratic form. We let $t(n)$ denote the number of representations of n by the ternary quadratic form $x^2 + 3y^2 + 3z^2$, so that

$$G(q) := \sum_{n=0}^{\infty} t(n)q^n := \sum_{x,y,z\in\mathbb{Z}} q^{x^2+3y^2+3z^2}.$$

Then we find

$$16c(n) = t(24n + 7),$$

for $n \geq 0$.

We need

Theorem 6.1 (Shemanske [13, Theorem 4.1])**.** *If n is square-free, and $n \equiv 7 \pmod{24}$, then we have*

$$t(n) = 16h(-n).$$

where $h(-n)$ is the class number of $\mathbb{Q}(\sqrt{-n})$.

Also we need analogs of of Theorem 2.2 and Theorem 2.3. Their proof follows similarly. Finally we need the following equation which is an analog of (4.1). We have

$$t(p^2n) + \left(\frac{-n}{p}\right) t(n) + pt(n/p^2) = (p + 1)t(n),$$

for $n \geq 0$ and any prime $p > 3$. This identity was conjectured by Cooper and Lam [7] and proved by Guo, Peng, and Qin [9]. In particular see [9, Conjecture 1.5].

We note that each of the eta-products $g(\tau)$ which occur in parts (i),(ii),(iii) respectively of Theorem 1.3 satisfy

$$\operatorname{ord}(g(\tau), \infty) = \frac{\ell}{24}$$

for $\ell = 11$, 19 and 7 respectively. It is natural to ask whether there are other weight 3/2 eta-products with similar mod 4 congruences where $(\ell, 24) = 1$ and $1 \leq \ell \leq 23$. We have found eta-products that satisfy congruences of the type in Theorem 1.3 for each of the remaining cases $\ell = 1, 5, 13, 17, 23$. However the nature of these congruences is more complicated. We will consider these other results in a later paper. It would be interesting if the methods of this paper can be applied to approach Conjectures 1.1 and 1.2.

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