CONGRUENCES MODULO POWERS OF 5 AND 7 FOR THE CRANK AND RANK PARITY FUNCTIONS AND RELATED MOCK THETA FUNCTIONS

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ABSTRACT. It is well known that Ramanujan conjectured congruences modulo powers of 5, 7 and and 11 for the partition function. These were subsequently proved by Watson (1938) and Atkin (1967). In 2009 Choi, Kang, and Lovejoy proved congruences modulo powers of 5 for the crank parity function. The generating function for the analogous rank parity function is $f(q)$, the first example of a mock theta function that Ramanujan mentioned in his last letter to Hardy. Recently we proved congruences modulo powers of 5 for the rank parity function, and here we extend these congruences for powers of 7. We also show how these congruences imply congruences modulo powers of 5 and 7 for the coefficients of the related third order mock theta function $\omega(q)$, using Atkin-Lehner involutions and transformation results of Zwegers. Finally we a prove a family of congruences modulo powers of 7 for the crank parity function.

1. INTRODUCTION

Let $p(n)$ be the number of unrestricted partitions of n. Ramanujan discovered and later proved that

$$p(5n+4) \equiv 0 \pmod{5}, \tag{1.1}$$

$$p(7n+5) \equiv 0 \pmod{7}, \tag{1.2}$$

$$p(11n+6) \equiv 0 \pmod{11}. \tag{1.3}$$

In 1944 Dyson [18] defined the *rank* of a partition as the largest part minus the number of parts and conjectured that residue of the rank mod 5 (resp. mod 7) divides the partitions of $5n+4$ (resp. $7n+5$) into 5 (resp. 7) equal classes thus giving combinatorial explanations of Ramanujan's partition congruences mod 5 and 7. Dyson's rank conjectures were proved by Atkin and Swinnerton-Dyer [7]. Dyson also conjectured the existence of another statistic

Date: July 6, 2024.

2020 *Mathematics Subject Classification.* 05A17, 11F33, 11F37, 11P83, 33D15.

Key words and phrases. partition congruences, Dyson's rank, mock theta functions, modular functions, non-holomorphic modular functions, Atkin-Lehner involutions.

The first author was supported in part by the National Natural Science Foundation of China (Grant No. 12201387) and Shanghai Sailing Program (#21YF1413600). The second author was supported in part by China Postdoctoral Science Foundation (#2022M712422). The third author was supported in part by a grant from the Simon's Foundation (#318714). The preliminary results of this paper were first presented at the AMS Special Session on Experimental and Computer Assisted Mathematics, Denver, January, 2020.

he called the *crank* which would likewise explain Ramanujan's partition congruence mod 11. The crank was found by Andrews and the third author [4] who defined the *crank* as the largest part, if the partition has no ones, and otherwise as the difference between the number of parts larger than the number of ones and the number of ones.

Let $M_e(n)$ (resp. $M_o(n)$) denote the number of partitions of n with even (resp. odd) crank. Choi, Kang and Lovejoy [15] proved congruences modulo powers of 5 the *crank parity function*, which is the difference

$$M_e(n) - M_o(n).$$

Theorem 1.1 (Choi, Kang and Lovejoy [15, Theorem 1.1])**.** *For all $\alpha \geq 0$ we have*

$$M_e(n) - M_o(n) \equiv 0 \pmod{5^{\alpha+1}}, \qquad \text{if } 24n \equiv 1 \pmod{5^{2\alpha+1}}.$$

This gave a weak refinement of Ramanujan's partition congruence modulo powers of 5:

$$p(n) \equiv 0 \pmod{5^a}, \qquad \text{if } 24n \equiv 1 \pmod{5^\alpha}.$$

Ramanujan's partition congruence was proved by Watson [31]. We define

$$\beta(n) = M_e(n) - M_o(n), \tag{1.4}$$

for $n \geq 0$. We prove the following new

Theorem 1.2. *For each $\alpha \geq 1$ there is an integral constant K_α such that*

$$\beta(49n-2) \equiv K_\alpha\, \beta(n) \pmod{7^\alpha}, \qquad \text{if } 24n \equiv 1 \pmod{7^\alpha}. \tag{1.5}$$

This gives a weak refinement of Ramanujan's partition congruence modulo powers of 7:

$$p(n) \equiv 0 \pmod{7^\alpha}, \qquad \text{if } 24n \equiv 1 \pmod{7^{\left\lfloor \frac{a+2}{2} \right\rfloor}}.$$

This was also proved by Watson [31]. The congruence in (1.5) is reminiscent of Atkin and O'Brien's [6] congruences mod powers of 13 for the partition function.

In [14] we considered analogues of Theorem 1.1 for the rank parity function. Analogous to $M_e(n)$ and $M_o(n)$ we let $N_e(n)$ (resp. $N_o(n)$) denote the number of partitions of n with even (resp. odd) rank. It is well known that the difference is related to Ramanujan's mock theta function $f(q)$. This is the first example of a mock theta function that Ramanujan gave in his last letter to Hardy. Let

$$\begin{aligned} f(q) &= \sum_{n=0}^{\infty} a_f(n) q^n = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2\cdots(1+q^n)^2} \\ &= 1 + q - 2\,q^2 + 3\,q^3 - 3\,q^4 + 3\,q^5 - 5\,q^6 + 7\,q^7 - 6\,q^8 + 6\,q^9 - 10\,q^{10} + 12\,q^{11} - 11\,q^{12} + \cdots. \end{aligned}$$

This function has been studied by many authors. Ramanujan conjectured an asymptotic formula for the coefficients $a_f(n)$. Dragonette [17] improved this result by finding a Rademacher-type asymptotic expansion for the coefficients. The error term was subsequently improved by Andrews [3], Bringmann and Ono [10], and Ahlgren and Dunn [1]. We have

$$a_f(n) = N_e(n) - N_o(n),$$

for $n \geq 0$.

In [14, Theorem 1.2], we proved

Theorem 1.3. *For all* $\alpha \geq 3$ *and all* $n \geq 0$ *we have*

$$a_f(5^\alpha n + \delta_\alpha) + a_f(5^{\alpha-2} n + \delta_{\alpha-2}) \equiv 0 \pmod{5^{\lfloor \frac{1}{2}\alpha \rfloor}}, \tag{1.6}$$

where δ_α *satisfies* $0 < \delta_\alpha < 5^\alpha$ *and* $24\delta_\alpha \equiv 1 \pmod{5^\alpha}$.

In [14], we also stated the following theorem without proof.

Theorem 1.4. *For all* $\alpha \geq 3$ *and all* $n \geq 0$ *we have*

$$a_f(7^\alpha n + \delta_\alpha) - a_f(7^{\alpha-2} n + \delta_{\alpha-2}) \equiv 0 \pmod{7^{\lfloor \frac{1}{2}(\alpha-1) \rfloor}}, \tag{1.7}$$

where δ_α *satisfies* $0 < \delta_\alpha < 7^\alpha$ *and* $24\delta_\alpha \equiv 1 \pmod{7^\alpha}$.

The starting point of our proof of Theorem 1.4 is an eta-product identity for the generating function of

$$a_f(n/7) - a_f(7n-2).$$

See Theorem 3.1. This enables us to use the theory of modular functions to prove the congruences. Our presentation and method are similar to that Paule and Radu [25], who solved a difficult conjecture of Sellers [29] for congruences modulo powers of 5 for Andrews's two-colored generalized Frobenius partitions [2].

The goal of this paper is prove Theorems 1.2 and 1.4, as well as prove analogous congruences for Ramanujan's third order mock theta function $\omega(q)$:

$$\begin{aligned} \omega(q) =& \sum_{n=0}^{\infty} a_w(n) q^n = 1 + \sum_{n=1}^{\infty} \frac{q^{2n^2+2n}}{(1-q)^2(1-q^3)^2 \cdots (1+q^{2n-1})^2} \\ =& 1 + q^4 + 2q^5 + 3q^6 + 4q^7 + 5q^8 + 6q^9 + 7q^{10} + 8q^{11} + 10q^{12} + \cdots . \end{aligned}$$

By utilising an Atkin-Lehner involution we show the following theorem follows from Theorems 1.3 and 1.4.

Theorem 1.5. (i) *For all* $\alpha \geq 3$ *and all* $n \geq 0$ *we have*

$$a_\omega(5^\alpha n + \delta_\alpha) + a_\omega(5^{\alpha-2} n + \delta_{\alpha-2}) \equiv 0 \pmod{5^{\lfloor \frac{1}{2}\alpha \rfloor}}, \tag{1.8}$$

where δ_α *satisfies* $0 < \delta_\alpha < 5^\alpha$ *and* $3\delta_\alpha + 2 \equiv 0 \pmod{5^\alpha}$.

(ii) *For all* $\alpha \geq 3$ *and all* $n \geq 0$ *we have*

$$a_\omega(7^\alpha n + \delta_\alpha) + a_\omega(7^{\alpha-2} n + \delta_{\alpha-2}) \equiv 0 \pmod{7^{\lfloor \frac{1}{2}(\alpha-1) \rfloor}}, \tag{1.9}$$

where δ_α *satisfies* $0 < \delta_\alpha < 7^\alpha$ *and* $3\delta_\alpha + 2 \equiv 0 \pmod{7^\alpha}$.

Remark. We note that Karl-Heine Fricke [19] independently observed (1.6)–(1.9) but without proof.

In Section 2 we include the necessary background and algorithms from the theory of modular functions for proving identities. In Sections 3–5 we apply the theory of modular functions to prove our Theorems 1.2, 1.4 and 1.5.

Some Remarks and Notation. Throughout this paper we use the standard q-notation. For finite products we use

$$(z;q)_n = (z)_n = \begin{cases} \prod_{j=0}^{n-1}(1-zq^j), & n>0 \\ 1, & n=0. \end{cases}$$

For infinite products we use

$$(z;q)_\infty = (z)_\infty = \lim_{n\to\infty}(z;q)_n = \prod_{n=1}^{\infty}(1-zq^{n-1}),$$

$$(z_1,z_2,\ldots,z_k;q)_\infty = (z_1;q)_\infty(z_2;q)_\infty\cdots(z_k;q)_\infty,$$

$$[z;q]_\infty = (z;q)_\infty(z^{-1}q;q)_\infty = \prod_{n=1}^{\infty}(1-zq^{n-1})(1-z^{-1}q^n),$$

$$[z_1,z_2,\ldots,z_k;q]_\infty = [z_1;q]_\infty[z_2;q]_\infty\cdots[z_k;q]_\infty,$$

for $|q|<1$ and z, z_1, z_2,..., $z_k \neq 0$. For θ-products we use

$$J_{a,b} = (q^a, q^{b-a}, q^b; q^b)_\infty \quad \text{and} \quad J_b = (q^b;q^b)_\infty,$$

and as usual

$$\eta(\tau) = \exp(\pi i\tau/12)\prod_{n=1}^{\infty}(1-\exp(2\pi in\tau)) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n), \tag{1.10}$$

where $\mathrm{Im}(\tau)>0$.

Throughout this paper we let $\lfloor x\rfloor$ denote the largest integer less than or equal to x, and let $\lceil x\rceil$ denote the smallest integer greater than or equal to x.

We need some notation for formal Laurent series. See the remarks at the end of [25, Section 1, p.823]. Let R be a ring and q be an indeterminant. We let $R((q))$ denote the formal Laurent series in q with coefficients in R. These are series of the form

$$f = \sum_{n\in\mathbb{Z}} a_n\, q^n,$$

where $a_n \neq 0$ for at most finitely many $n<0$. For $f\neq 0$ we define the order of f (with respect to q) as the smallest integer N such that $a_N\neq 0$ and write $N=\mathrm{ord}_q(f)$. We note that if f is a modular function this coincides with $\mathrm{ord}(f,\infty)$. See equation (2.1) below for this other notation. Suppose t and $f \in R((q))$ and the composition $f \circ t$ is well-defined as a formal Laurent series. This is the case if $\mathrm{ord}_q(t) > 0$. The t-order of

$$F = f \circ t = \sum_{n \in \mathbb{Z}} a_n \, t^n,$$

where $t = \sum_{n \in \mathbb{Z}} b_n \, q^n$, is defined to be the smallest integer N such that $a_N \neq 0$ and write $N = \mathrm{ord}_t(F)$. For example, if

$$f = q^{-1} + 1 + 2\,q + \cdots, \qquad t = q^2 + 3q^3 + 5q^4 + \cdots,$$

then

$$F = f \circ t = t^{-1} + 1 + 2\,t + \cdots, \qquad = q^{-2} - 3q^{-1} + 5 + \cdots,$$

so that $\mathrm{ord}_q(f) = -1$, $\mathrm{ord}_q(t) = 2$, $\mathrm{ord}_t(F) = -1$ and $\mathrm{ord}_q(F) = -2$.

2. Modular Functions

In this section we present the needed theory of modular functions which we use to prove identities. A general reference is Rankin's book [26].

2.1. **Background theory.** Our presentation is based on [8, pp.326-329]. Let $\mathscr{H} = \{\tau : \mathrm{Im}(\tau) > 0\}$ (the complex upper half-plane). For each $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2^+(\mathbb{Z})$, the set of integer 2×2 matrix with positive determinant, the bilinear transformation $M(\tau)$ is defined by

$$M\tau = M(\tau) = \frac{a\tau + b}{c\tau + d}.$$

The stroke operator is defined by

$$(f \mid M)(\tau) = f(M\tau),$$

and satisfies

$$f \mid MS = f \mid M \mid S.$$

The modular group $\Gamma(1)$ is defined by

$$\Gamma(1) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2^+(\mathbb{Z}) \,:\, ad - bc = 1 \right\}.$$

We consider the following subgroups Γ of the modular group with finite index

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) \,:\, c \equiv 0 \pmod{N} \right\},$$

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) \,:\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}.$$

Such a group Γ acts on $\mathscr{H} \cup \mathbb{Q} \cup \{\infty\}$ by the transformation $V(\tau)$, for $V \in \Gamma$ which induces an equivalence relation. We call a set $\mathscr{F} \subseteq \mathscr{H} \cup \mathbb{Q} \cup \{\infty\}$ a *fundamental set* for Γ if it contains one element of each equivalence class. The finite set $\mathcal{F} \cap (\mathbb{Q} \cup \{\infty\})$ is called the *complete set of inequivalent cusps*.

A function $f : \mathscr{H} \longrightarrow \mathbb{C}$ is a *weakly holomorphic modular function* on Γ if the following conditions hold:

(i) f is holomorphic on $\mathscr{H}$.
(ii) $f \mid V = f$ for all $V \in \Gamma$.
(iii) For every $A \in \Gamma(1)$ the function $f \mid A^{-1}$ has an expansion

$$(f \mid A^{-1})(\tau) = \sum_{m=m_0}^{\infty} b(m) \exp(2\pi i \tau m/\kappa)$$

on some half-plane $\{\tau : \operatorname{Im}\tau > h \geq 0\}$, where $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and

$$\kappa = \min\left\{k > 0 : \pm A^{-1} T^k A \in \Gamma\right\}.$$

The positive integer $\kappa = \kappa(\Gamma; \zeta)$ is called the *fan width* of Γ at the cusp $\zeta = A^{-1}\infty$. If $b(m_0) \neq 0$, then we write

$$\operatorname{Ord}(f, \zeta, \Gamma) = m_0$$

which is called the *order* of f at ζ with respect to Γ. We also write

$$\operatorname{ord}(f; \zeta) = \frac{m_0}{\kappa} = \frac{m_0}{\kappa(\Gamma, \zeta)}, \tag{2.1}$$

which is called the *invariant order* of f at ζ. For each $z \in \mathscr{H}$, $\operatorname{ord}(f; z)$ denotes the order of f at z as an analytic function of z, and the order of f with respect to Γ is defined by

$$\operatorname{Ord}(f, z, \Gamma) = \frac{1}{\ell} \operatorname{ord}(f; z)$$

where ℓ is the order of z as a fixed point of Γ. We note $\ell = 1$, 2 or 3. Our main tool for proving modular function identities is the valence formula [26, Theorem 4.1.4, p.98]. If $f \neq 0$ is a modular function on Γ and $\mathscr{F}$ is any fundamental set for Γ then

$$\sum_{z \in \mathscr{F}} \operatorname{Ord}(f, z, \Gamma) = 0. \tag{2.2}$$

2.2. **Eta-product identities.** We will consider eta-products of the form

$$f(\tau) = \prod_{d \mid N} \eta(d\tau)^{m_d}, \tag{2.3}$$

where N is a positive integer, each $d > 0$ and $m_d \in \mathbb{Z}$.

Modularity. Newman [24] has found necessary and sufficient conditions under which an eta-product is a modular function on $\Gamma_0(N)$.

Theorem 2.1 ([24, Theorem 4.7]). *The function $f(\tau)$ (given in (2.3)) is a modular function on $\Gamma_0(N)$ if and only if*

(1) $\sum_{d|N} m_d = 0,$

(2) $\sum_{d|N} d m_d \equiv 0 \pmod{24},$

(3) $\sum_{d|N} \frac{N m_d}{d} \equiv 0 \pmod{24},$ *and*

(4) $\prod_{d|N} d^{|m_d|}$ *is a square.*

Orders at cusps. Ligozat [22] has computed the invariant order of an eta-product at the cusps of $\Gamma_0(N)$.

Theorem 2.2 ([22, Theorem 4.8]). *If the eta-product $f(\tau)$ (given in (2.3)) is a modular function on $\Gamma_0(N)$, then its order at the cusp $\zeta = \frac{b}{c}$ (assuming $(b,c)=1$) is*

$$\operatorname{ord}(f(\tau);\zeta) = \sum_{d|N} \frac{(d,c)^2 m_d}{24d}. \tag{2.4}$$

Chua and Lang [16] have found a set of inequivalent cusps for $\Gamma_0(N)$.

Theorem 2.3 ([16, p.354]). *Let N be a positive integer and for each positive divisor d of N let $e_d = (d, N/d)$. Then the set*

$$\Delta = \underset{d|N}{\cup} S_d$$

is a complete set of inequivalent cusps of $\Gamma_0(N)$ where

$$S_d = \{x_i/d \,:\, (x_i,d)=1, \quad 0 \le x_i \le d-1, \quad x_i \not\equiv x_j \pmod{e_d}\}.$$

Biagioli [9] has found the fan width of the cusps of $\Gamma_0(N)$.

Lemma 2.4 ([9, Lemma 4.2]). *If $(r,s)=1$, then the fan width of $\Gamma_0(N)$ at $\frac{r}{s}$ is*

$$\kappa\left(\Gamma_0(N); \frac{r}{s}\right) = \frac{N}{(N,s^2)}.$$

An application of the valence formula. Since eta-products have no zeros or poles in $\mathscr{H}$ the following result follows easily from the valence formula (2.2).

Theorem 2.5. *Let $f_1(\tau)$, $f_2(\tau)$, $\ldots$, $f_n(\tau)$ be eta-products that are modular functions on $\Gamma_0(N)$. Let $\mathcal{S}_N$ be a set of inequivalent cusps for $\Gamma_0(N)$. Define the constant*

$$B = \sum_{\substack{\zeta \in \mathcal{S}_N \\ \zeta \neq \infty}} min(\{\mathrm{Ord}(f_j, \zeta, \Gamma_0(N)) \,:\, 1 \le j \le n\}), \tag{2.5}$$

and consider

$$g(\tau) := \alpha_1 f_1(\tau) + \alpha_2 f_2(\tau) + \cdots + \alpha_n f_n(\tau), \tag{2.6}$$

where each $\alpha_j \in \mathbb{C}$. Then

$$g(\tau) \equiv 0$$

if and only if

$$\mathrm{Ord}(g(\tau), \infty, \Gamma_0(N)) > -B. \tag{2.7}$$

An algorithm for proving eta-product identities.

STEP 0. Write the identity in the following form:

$$\alpha_1 f_1(\tau) + \alpha_2 f_2(\tau) + \cdots + \alpha_n f_n(\tau) = 0, \tag{2.8}$$

where each $\alpha_i \in \mathbb{C}$ and each $f_i(\tau)$ is an eta-product of level N.

STEP 1. Use Theorem 2.1 to check that $f_j(\tau)$ is a modular function on $\Gamma_0(N)$ for each $1 \le j \le n$.

STEP 2. Use Theorem 2.3 to find a set $\mathcal{S}_N$ of inequivalent cusps for $\Gamma_0(N)$ and the fan width of each cusp.

STEP 3. Use Theorem 2.2 to calculate the order of each eta-product $f_j(\tau)$ at each cusp of $\Gamma_0(N)$.

STEP 4. Calculate

$$B = \sum_{\substack{\zeta \in \mathcal{S}_N \\ \zeta \neq \infty}} \min(\{\mathrm{Ord}(f_j, \zeta, \Gamma_0(N)) \,:\, 1 \le j \le n\}).$$

STEP 5. Show that

$$\mathrm{Ord}(g(\tau), \infty, \Gamma_0(N)) > -B$$

where

$$g(\tau) = \alpha_1 f_1(\tau) + \alpha_2 f_2(\tau) + \cdots + \alpha_n f_n(\tau).$$

Theorem 2.5 then implies that $g(\tau) \equiv 0$ and hence the eta-product identity (2.8).

The third author has written a MAPLE package called `ETA` which implements this algorithm. See

`http://qseries.org/fgarvan/qmaple/ETA/`

A modular equation. Define

$$t := t(\tau) := \frac{\eta(7\tau)^4}{\eta(\tau)^4}. \tag{2.9}$$

We note that $t(\tau)$ is a Hauptmodul for $\Gamma_0(7)$ [23]. As an application of our algorithm we prove the following theorem which will be needed later.

Theorem 2.6. *Let*

$$a_0(t) = t, \tag{2.10}$$
$$a_1(t) = 7^2t^2 + 4\cdot 7t, \tag{2.11}$$
$$a_2(t) = 7^4t^3 + 4\cdot 7^3t^2 + 46\cdot 7t, \tag{2.12}$$
$$a_3(t) = 7^6t^4 + 4\cdot 7^5t^3 + 46\cdot 7^3t^2 + 272\cdot 7t, \tag{2.13}$$
$$a_4(t) = 7^8t^5 + 4\cdot 7^7t^4 + 46\cdot 7^5t^3 + 272\cdot 7^3t^2 + 845\cdot 7t, \tag{2.14}$$
$$a_5(t) = 7^{10}t^6 + 4\cdot 7^9t^5 + 46\cdot 7^7t^4 + 272\cdot 7^5t^3 + 845\cdot 7^3t^2 + 176\cdot 7^2t, \tag{2.15}$$
$$a_6(t) = 7^{12}t^7 + 4\cdot 7^{11}t^6 + 46\cdot 7^9t^5 + 272\cdot 7^7t^4 + 845\cdot 7^5t^3 + 176\cdot 7^4t^2 + 82\cdot 7^2t. \tag{2.16}$$

where $t = t(\tau)$ is defined in (2.9). Then

$$t(\tau)^7 - \sum_{l=0}^{6} a_l(t(7\tau))t(\tau)^l = 0. \tag{2.17}$$

Proof. From Theorem 2.1 we find that $t(\tau)$ is a modular function on $\Gamma_0(7)$ and $t(7\tau)$ is a modular function on $\Gamma_0(49)$. Hence each term on the left side of (2.17) is a modular function on $\Gamma_0(49)$. For convenience we divide by $t(\tau)^7$ and let

$$g(\tau) = 1 - \sum_{l=0}^{6} a_l(t(7\tau))t(\tau)^{l-7}. \tag{2.18}$$

From Theorem 2.3, Lemma 2.4 and Theorem 2.2 we have the following table of fan widths for the cusps of $\Gamma_0(49)$, with the orders and invariant orders of both $t(\tau)$ and $t(7\tau)$.

ζ	0	1/7	2/7	3/7	4/7	5/7	6/7	1/49
$\kappa(\Gamma_0(49),\zeta)$	49	1	1	1	1	1	1	1
$\mathrm{ord}(t(\tau),\zeta)$	$-1/7$	1	1	1	1	1	1	1
$\mathrm{Ord}(t(\tau),\zeta,\Gamma_0(49))$	-7	1	1	1	1	1	1	1
$\mathrm{ord}(t(7\tau),\zeta)$	$-1/49$	-1	-1	-1	-1	-1	-1	7
$\mathrm{Ord}(t(7\tau),\zeta,\Gamma_0(49))$	-1	-1	-1	-1	-1	-1	-1	7

Expanding the right side of (2.18) gives 29 terms of the form $t(7\tau)^k t(\tau)^{j-7}$ with $1 \le k \le j+1$ where $0 \le j \le 6$, together with $(k,j) = (0,7)$. We calculate the order of each term at each cusp ζ of $\Gamma_0(49)$, and thus giving lower bounds for $\mathrm{Ord}(g(\tau),\zeta,\Gamma_0(49))$.

ζ	0	1/7	2/7	3/7	4/7	5/7	6/7	1/49
$\mathrm{Ord}(g(\tau),\zeta,\Gamma_0(49)) \ge$	0	−8	−8	−8	−8	−8	−8	0

Thus the constant B in Theorem 2.5 is $B = -48$. It suffices to show that

$$\mathrm{Ord}(g(\tau),\infty,\Gamma_0(49)) > 48.$$

This is easily verified. Thus by Theorem 2.5 we have $g(\tau) \equiv 0$ and the result follows. □

2.3. **The U_p operator.** Let p prime and

$$f = \sum_{m=m_0}^{\infty} a(m) q^m$$

be a formal Laurent series. We define U_p by

$$U_p(f) := \sum_{pm \ge m_0} a(pm) q^m. \tag{2.19}$$

If f and h are modular functions (with $q = \exp(2\pi i\tau)$),

$$U_p(f) = \frac{1}{p}\sum_{j=0}^{p} f \;\Big|\; \begin{pmatrix} 1/p & j/p \\ 0 & 1 \end{pmatrix} = \frac{1}{p}\sum_{j=0}^{p} f\left(\frac{\tau+j}{p}\right), \tag{2.20}$$

and for

$$H(\tau) = h(p\tau),$$

we have

$$U_p(fH)(\tau) = h(\tau) U_p(f)(\tau). \tag{2.21}$$

Theorem 2.7 ([5, Lemma 7, p.138])**.** *Let p be prime. If f is a modular function on $\Gamma_0(pN)$ and $p \mid N$, then $U_p(f)$ is a modular function on $\Gamma_0(N)$.*

3. The rank parity function modulo powers of 7

3.1. **A Generating Function.** In this section we prove an identity for the generating function of

$$a_f(n/7) - a_f(7n-2),$$

where it is understood that $a_f(n) = 0$ if n is not a non-negative integer.

Theorem 3.1. *We have*

$$f(q^7) - q^2 f_5(q) = \frac{J_7^3}{J_2^2}\left(\frac{J_1^3 J_7^3}{J_2^3 J_{14}^3} + 6q^2 \frac{J_{14}^4 J_1^4}{J_2^4 J_7^4}\right). \tag{3.1}$$

Remark. We note that this theorem can also be proved from Theorem 5.6.

Proof. From Watson [30, p.64] we have

$$f(q)=\frac{2}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n(3n+1)/2}}{1+q^n}. \tag{3.2}$$

We find that

$$\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n(3n+1)/2+4n}}{1+q^{7n}}=\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n(3n+1)/2+2n}}{1+q^{7n}},$$
$$\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n(3n+1)/2+5n}}{1+q^{7n}}=\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n(3n+1)/2+n}}{1+q^{7n}},$$
$$\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n(3n+1)/2+6n}}{1+q^{7n}}=\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n(3n+1)/2}}{1+q^{7n}}.$$

By [13, Theorem 2.1] we have

$$\begin{aligned}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n(3n+1)/2}}{1+q^{7n}}=&-P(q^7,-q^7;q^{49})+q^{-6}P(q^{14},-q^7;q^{49})-q^{-9}P(q^{21},-q^7;q^{49})\\&+\frac{J_1}{J_{49}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{(147n^2+49n)/2-7}}{1+q^{49n-7}},\end{aligned} \tag{3.3}$$

$$\begin{aligned}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n(3n+1)/2+n}}{1+q^{7n}}=&P(q^{21},-q^{21};q^{49})-q^3P(q^{14},-q^{28};q^{49})+q^9P(q^7,-q^{28};q^{49})\\&-\frac{J_1}{J_{49}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{(147n^2-7n)/2-7}}{1+q^{49n-14}},\end{aligned} \tag{3.4}$$

$$\begin{aligned}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n(3n+1)/2+2n}}{1+q^{7n}}=&P(q^{14},-q^{14};q^{49})-q^6P(q^7,-q^{14};q^{49})-q^{-3}P(q^{21},-q^{14};q^{49})\\&-\frac{J_1}{J_{49}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{(147n^2+147n)/2+13}}{1+q^{49n+14}},\end{aligned} \tag{3.5}$$

$$\begin{aligned}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n(3n+1)/2+3n}}{1+q^{7n}}=&q^{-9}P(q^7,-1;q^{49})-q^{-15}P(q^{14},-1;q^{49})+q^{-18}P(q^{21},-1;q^{49})\\&-\frac{J_1}{J_{49}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{(147n^2+49n)/2-2}}{1+q^{49n}},\end{aligned} \tag{3.6}$$

where

$$P(a,b;q)=\frac{[a,a^2;q]_\infty (q;q)_\infty^2}{[b/a,ab,b;q]_\infty}. \tag{3.7}$$

From (3.2)-(3.6), and noting that $P(q^7,-q^7;q^{49})=P(q^{14},-q^{14};q^{49})$ we have

$$\begin{aligned}
f(q)=&\frac{2}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n(3n+1)/2}}{1+q^n}\\
=&\frac{2}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n(3n+1)/2}(1-q^n+q^{2n}-q^{3n}+q^{4n}-q^{5n}+q^{6n})}{1+q^{7n}}\\
=&\frac{2}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n(3n+1)/2}(2-2q^n+2q^{2n}-q^{3n})}{1+q^{7n}}\\
=&\frac{2}{J_1}(2q^{-6}P(q^{14},-q^7;q^{49})-2q^{-9}P(q^{21},-q^7;q^{49})+2q^3P(q^{14},-q^{28};q^{49})\\
&-2q^9P(q^7,-q^{28};q^{49})-2q^6P(q^7,-q^{14};q^{49})-2q^{-3}P(q^{21},-q^{14};q^{49})\\
&-q^{-9}P(q^7,-1;q^{49})+q^{-15}P(q^{14},-1;q^{49})-q^{-18}P(q^{21},-1;q^{49})-\frac{J_7^4}{J_{14}^2})\\
&+\frac{4}{J_{49}}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{(147n^2+49n)/2-7}}{1+q^{49n-7}}+\frac{4}{J_{49}}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{(147n^2-7n)/2-7}}{1+q^{49n-14}}\\
&-\frac{4}{J_{49}}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{(147n^2+147n)/2+13}}{1+q^{49n+14}}+\frac{1}{q^2}f(q^{49}).
\end{aligned} \tag{3.8}$$

We let

$$\begin{aligned}
g(q):=&\frac{2}{J_1}(2q^{-6}P(q^{14},-q^7;q^{49})-2q^{-9}P(q^{21},-q^7;q^{49})+2q^3P(q^{14},-q^{28};q^{49})\\
&-2q^9P(q^7,-q^{28};q^{49})-2q^6P(q^7,-q^{14};q^{49})-2q^{-3}P(q^{21},-q^{14};q^{49})\\
&-q^{-9}P(q^7,-1;q^{49})+q^{-15}P(q^{14},-1;q^{49})-q^{-18}P(q^{21},-1;q^{49})-\frac{J_7^4}{J_{14}^2}),
\end{aligned} \tag{3.9}$$

write the 7-dissection of $g(q)$ as

$$g(q)=g_0(q^7)+q\,g_1(q^7)+\cdots+q^6\,g_6(q^7). \tag{3.10}$$

From (3.2), (3.8) and (3.10), replacing q^7 by q, we have

$$\sum_{n=0}^{\infty}a_f(7n+5)q^n=\frac{1}{q^2}f(q^7)+g_5(q) \tag{3.11}$$

after dividing both sides by q^5 and replacing q^7 by q.

The 7-dissection of J_1 is well-known

$$J_1 = J_{49} \times \left(A(q^7) - q - B(q^7)q^2 + \frac{q^5}{A(q^7)B(q^7)} \right), \tag{3.12}$$

where

$$A(q) := \frac{J_{2,7}}{J_{1,7}}, \quad B(q) := \frac{J_{3,7}}{J_{2,7}}.$$

See for example [20, Lemma 3.18].

From (3.9), (3.11) and (3.12),

$$\begin{aligned} &J_{49}(g_0(q^7) + qg_1(q^7) + \cdots + q^6 g_6(q^7)) \left(A(q^7) - qB(q^7) - q^2 + \frac{q^5}{A(q^7)B(q^7)} \right) \\ =&2(2q^{-6}P(q^{14}, -q^7; q^{49}) - 2q^{-9}P(q^{21}, -q^7; q^{49}) + 2q^3 P(q^{14}, -q^{28}; q^{49}) \\ &- 2q^9 P(q^7, -q^{28}; q^{49}) - 2q^6 P(q^7, -q^{14}; q^{49}) - 2q^{-3} P(q^{21}, -q^{14}; q^{49}) \\ &- q^{-9} P(q^7, -1; q^{49}) + q^{-15} P(q^{14}, -1; q^{49}) - q^{-18} P(q^{21}, -1; q^{49}) - \frac{J_7^4}{J_{14}^2}). \end{aligned} \tag{3.13}$$

By expanding the left side of (3.13) and comparing both sides according to the residue of the exponent of q modulo 7, we obtain 7 equations:

$$A(q^7)g_0 + \frac{q^7 g_2}{A(q^7)B(q^7)} - q^7 g_5 - q^7 B(q^7) g_6 = \frac{2J_7^4}{J_{14}^2 J_{49}}, \tag{3.14}$$

$$-B(q^7)g_0 + A(q^7)g_1 + \frac{q^7 g_3}{A(q^7)B(q^7)} - q^7 g_6 = \frac{4P(q^{14}, -q^7)}{q^7 J_{49}}, \tag{3.15}$$

$$g_0 + B(q^7)g_1 - A(q^7)g_2 - \frac{q^7 g_4}{A(q^7)B(q^7)} = \frac{4q^7 P(q^7, -q^{21})}{J_{49}}, \tag{3.16}$$

$$g_1 + B(q^7)g_2 - A(q^7)g_3 - \frac{q^7 g_5}{A(q^7)B(q^7)} = \frac{2P(q^{21}, -1) - 4q^{21}P(q^{14}, -q^{21})}{q^{21} J_{49}}, \tag{3.17}$$

$$g_2 + B(q^7)g_3 - A(q^7)g_4 - \frac{q^7 g_6}{A(q^7)B(q^7)} = \frac{4P(q^{21}, -q^{14})}{q^7 J_{49}}, \tag{3.18}$$

$$\frac{-g_0}{A(q^7)B(q^7)} + g_3 + B(q^7)g_4 - A(q^7)g_5 = \frac{2P(q^7, -1) + 4P(q^{21}, -q^7)}{q^{14} J_{49}}, \tag{3.19}$$

$$\frac{g_1}{A(q^7)B(q^7)} - g_4 - B(q^7)g_5 + A(q^7)g_6 = \frac{2P(q^{14}, -1) - 4q^{21}P(q^7, -q^{14})}{q^{21} J_{49}}. \tag{3.20}$$

where $g_j = g_j(q^7)$ for $0 \le j \le 6$.

Solving these equations we find that

$$\begin{aligned} g_5(q) =& \frac{1}{H} \{ (A^6 B^9 + 4A^7 B^7 + 3A^8 B^5 - A^9 B^3 + 3A^4 B^6 q + 8A^5 B^4 q \\ &- 4A^6 B^2 q - 4A^2 B^3 q^2 - 3A^3 B q^2 + q^3) A^2 B^3 X_0 \end{aligned}$$

$$
\begin{aligned}
&+ (A^6B^9 + 3A^7B^7 + A^8B^5 - 2A^3B^8q - 3A^4B^6q + 6A^5B^4q\\
&+ AB^5q^2 + 2A^2B^3q^2 - q^3)A^3B^2X_1\\
&+ (A^7B^7 + 2A^8B^5 + A^3B^8q + 2A^4B^6q + 3A^5B^4q + A^6B^2q\\
&- 6A^2B^3q^2 + 3A^3Bq^2 + q^3)A^3B^3X_2\\
&+ (A^{10}B^8 + A^{11}B^6 + A^6B^9q + 4A^8B^5q + 6A^4B^6q^2\\
&- A^5B^4q^2 - 5A^2B^3q^3 + q^4)ABX_3\\
&+ (A^9B^5 + A^4B^8q + 6A^5B^6q + 2A^6B^4q - 3A^2B^5q^2 + 3A^3B^3q^2\\
&- A^4Bq^2 + B^2q^3 - 2Aq^3)A^3B^3X_4\\
&+ (A^9B^3 - A^3B^8q - 4A^4B^6q + A^5B^4q + 5A^6B^2q\\
&+ AB^5q^2 + 6A^3Bq^2 - q^3)A^4B^4X_5\\
&+ (A^5B^{11} + 5A^6B^9 + 6A^7B^7 - A^8B^5 - A^4B^6q\\
&- 4A^2B^3q^2 + A^3Bq^2 + q^3)qA^2B^2X_6\big\},
\end{aligned}
$$

where $A := A(q)$, $B := B(q)$,

$$
\begin{aligned}
H :=& - A^7B^{14}q + A^{14}B^7 - 7A^8B^{12}q - 14A^9B^{10}q + 7A^{11}B^6q - 8A^7B^7q^2\\
&+ 14A^8B^5q^2 + 14A^4B^6q^3 - 7A^2B^3q^4 + q^5,
\end{aligned}
$$

and $X_0 - X_6$ are the right sides of (3.14) – (3.20) (respectively) after replacing q^7 by q. Then using the third author's `thetaids` MAPLE package, see

`http://qseries.org/fgarvan/qmaple/thetaids/`

we can prove,

$$
H = \frac{J_1^8}{J_7^8}A^7B^7, \tag{3.21}
$$

and then

$$
g_5(q) = -\frac{J_7^3}{J_2^2}\left(\frac{J_1^3J_7^3}{q^2J_2^3J_{14}^3} + 6\frac{J_1^4J_{14}^4}{J_2^4J_7^4}\right). \tag{3.22}
$$

From (3.11) and (3.22) we have

$$
f(q^7) - \sum_{n=0}^{\infty} a_f(7n-2)q^n = q^2g_5(q) = \frac{J_7^3}{J_2^2}\left(\frac{J_1^3J_7^3}{J_2^3J_{14}^3} + 6q^2\frac{J_1^4J_{14}^4}{J_2^4J_7^4}\right),
$$

which is our result (3.1). □

3.2. **A Fundamental Lemma.** We need the following fundamental lemma, whose proof follows easily from Theorem 2.6.

Lemma 3.2 (A Fundamental Lemma). *Suppose $u = u(\tau)$, and j is any integer. Then*

$$U_7(u\,t^j) = \sum_{l=0}^{6} a_l(\tau)\, U_7(u\,t^{j+l-7}),$$

where $t = t(\tau)$ is defined in (2.9) and the $a_j(\tau)$ are given in (2.10)–(2.16).

Proof. The result follows easily from (2.17) by multiplying both sides by $u\,t^{j-7}$ and applying U_7. □

We can check for each $a_j(t)$ that there exist integers $s(j,l)$ satisfying

$$a_j(t) = \sum_{l=1}^{7} s(j,l) 7^{[(7l+j-4)/4]} t^l. \tag{3.23}$$

Let $g = \sum_n a_n t^n, g \neq 0$, be such that $a_n = 0$ for almost all $n < 0$. Then the order of g is the smallest integer N such that $a_N \neq 0$, and we write $N = ord_t(g)$.

Lemma 3.3. *Let $u, v_1, v_2, v_3 : \mathbb{H} \to \mathbb{C}$ and $l \in \mathbb{Z}$. Suppose for $l \le k \le l+6$ and $i = 1, 2, 3$ there exist Laurent polynomials $p_k^{(i)}(t) \in \mathbb{Z}[t, t^{-1}]$ such that*

$$U_7(ut^k) = v_1 p_k^{(1)}(t) + v_2 p_k^{(2)}(t) + v_3 p_k^{(3)}(t), \tag{3.24}$$

and

$$ord_t(p_k^{(i)}(t)) \ge \left[\frac{k+s_i}{7}\right], \tag{3.25}$$

for fixed integers s_i. Then there exist families of Laurent polynomials $p_k^{(i)}(t) \in \mathbb{Z}[t, t^{-1}]$, $k \in \mathbb{Z}$, such that (3.24) and (3.25) hold for all $k \in \mathbb{Z}$.

Proof. Let $N > l+6$ be an integer and assume by induction that there are families of Laurent polynomials $p_k^{(i)}(t)$, $i \in 1, 2, 3$, such that (3.24) and (3.25) hold for $l \le k \le N-1$. Suppose

$$p_k^{(i)}(t) = \sum_{n \ge [(k+s_i)/7]} c_i(k,n) t^n,\ 1 \le k \le N-1,$$

with integers $c_i(k,n)$. Applying Lemma 3.2 we obtain:

$$\begin{aligned} U_7(ut^N) &= \sum_{j=0}^{6} a_j(t) U_7(ut^{N+j-7}) \\ &= \sum_{j=0}^{6} a_j(t) \sum_{i=1}^{3} v_i \sum_{n \ge [(N+j-7+s_i)/7]} c_i(N+j-7,n) t^n \end{aligned}$$

$$= \sum_{i=1}^{3} v_i \sum_{j=0}^{6} a_j(t)t^{-1} \sum_{n \geq [(N+j+s_i)/7]} c_i(N+j-7, n-1)t^n.$$

Recalling the fact that $a_j(t)t^{-1}$ for $0 \leq j \leq 6$ is a polynomial of t, this determines Laurent polynomials $P_N^{(i)}(t)$ with the desired properties. The induction proof for $N < l$ is analogous. □

Lemma 3.4. *Let $u, v_1, v_2, v_3 : \mathbb{H} \to \mathbb{C}$ and $l \in \mathbb{Z}$. Suppose for $l \leq k \leq l+6$ and $i = 1, 2, 3$ there exist Laurent polynomials $p_k^{(i)}(t) \in \mathbb{Z}[t, t^{-1}]$ such that*

$$U_7(ut^k) = v_1 p_k^{(1)}(t) + v_2 p_k^{(2)}(t) + v_3 p_k^{(3)}(t), \tag{3.26}$$

where

$$p_k^{(i)}(t) = \sum_n c_i(k, n) 7^{[\frac{7n-k+r_i}{4}]} t^n, \tag{3.27}$$

with integers r_i and $c_i(k, n)$. Then there exist families of Laurent polynomials $p_k^{(i)}(t) \in \mathbb{Z}[t, t^{-1}]$, $k \in \mathbb{Z}$, of the form (3.27) for which property (3.26) holds for all $k \in \mathbb{Z}$.

Proof. Suppose for an integer $N > l+6$ there are families of Laurent polynomials $p_k^{(i)}(t)$, $i \in 1, 2, 3$, of the form (3.27) satisfying property (3.26) for $l \leq k \leq N-1$. We proceed by mathematical induction on N. Applying Lemma 3.2 and using the induction base (3.26) and (3.27) we obtain:

$$U_7(ut^N) = \sum_{j=0}^{6} a_j(t) \sum_{i=1}^{3} v_i \sum_n c_i(N+j-7, n) 7^{[\frac{7n-(N+j-7)+r_i}{4}]} t^n.$$

Utilizing (3.23) Lemma 3.2 we have

$$U_7(ut^N) = \sum_{j=0}^{6} \sum_{l=1}^{7} s(j,l) 7^{[\frac{7l+j-4}{4}]} t^l \sum_{i=1}^{3} v_i \sum_n c_i(N+j-7, n) 7^{[\frac{7n-(N+j-7)+r_i}{4}]} t^n$$
$$= \sum_{i=1}^{3} v_i \sum_{j=0}^{6} \sum_{l=1}^{7} \sum_n s(j,l) c_i(N+j-7, n-l) 7^{[\frac{7(n-l)-(N+j-7)+r_i}{4}]+[\frac{7l+j-4}{4}]} t^n.$$

The induction step is completed by simplifying the exponent of 7 as follows:

$$\left[\frac{7(n-l)-(N+j-7)+r_i}{4}\right] + \left[\frac{7l+j-4}{4}\right]$$
$$\geq \left[\frac{7(n-l)-(N+j-7)+r_i+7l+j-4-3}{4}\right]$$
$$= \left[\frac{7n-N+r_i}{4}\right].$$

The induction proof for $N < l$ is analogous. □

3.3. **Proof of Theorem 1.4.** The proof depends on the forty-two fundamental relations listed in the Appendix A. These identities can be proved using the algorithm described in [14, Section 2C, pp.8-9]. From Theorem 3.1 we have

$$\sum_{n=0}^{\infty}(a_f(n/7)-a_f(7n-2))q^n=\frac{J_7^3}{J_2^2}\left(\frac{J_1^3J_7^3}{J_2^3J_{14}^3}+6q^2\frac{J_{14}^4J_1^4}{J_2^4J_7^4}\right). \tag{3.28}$$

For $f:\mathbb{H}\to\mathbb{C}$ we define $U_A(f)$ and $U_B(f):\mathbb{H}\to\mathbb{C}$ by

$$U_A(f):=U_7(Af),\quad U_B(f):=U_7(Bf),$$

where

$$A:=\frac{q^8J_{98}^2}{J_2^2},\quad B:=\frac{J_1^3}{q^6J_{49}^3}.$$

Define

$$L_0:=7p_0+p_1,$$

and for $\alpha\geq 0$ define

$$L_{2\alpha+1}=U_A(L_{2\alpha}),\quad L_{2\alpha+2}=U_B(L_{2\alpha+1}),$$

where

$$p_0:=\frac{qJ_{14}^4J_1^4}{J_7^4J_2^4},\quad p_1:=\frac{J_1^3J_7^3}{qJ_2^3J_{14}^3}-p_0.$$

Using (2.19), (2.21) and (3.28), it is easy to verify that for $\alpha\geq 0$ we have

$$L_{2\alpha}=\frac{J_2^2}{qJ_7^3}\sum_{n=0}^{\infty}A(7^{2\alpha}n+\lambda_{2\alpha})q^n,$$

$$L_{2\alpha+1}=\frac{qJ_{14}^2}{J_1^3}\sum_{n=0}^{\infty}A(7^{2\alpha+1}n+\lambda_{2\alpha+1})q^n,$$

where

$$A(n):=a_f(n/7)-a_f(7n-2)$$

and

$$\lambda_{2\alpha}=\lambda_{2\alpha+1}=\frac{7}{24}(1-7^{2\alpha}).$$

Following [25] we call a map $a:\mathbb{Z}\longrightarrow\mathbb{Z}$ a *discrete function* if it has finite support. We define

$$X_A$$
$$:=\left\{\sum_{k=0}^{\infty}r_1(k)7^{[\frac{7k}{4}]}t^k+p_0\sum_{k=0}^{\infty}r_2(k)7^{[\frac{7k}{4}]}t^k+p_1\sum_{k=1}^{\infty}r_3(k)7^{[\frac{7k-3}{4}]}t^k\ :\ \text{each } r_j \text{ is a discrete function}\right\},$$

$$X_B$$
$$:=\left\{\sum_{k=1}^{\infty}r_1(k)7^{[\frac{7k-5}{4}]}t^k+p_0\sum_{k=1}^{\infty}r_2(k)7^{[\frac{7k-5}{4}]}t^k+p_1\sum_{k=2}^{\infty}r_3(k)7^{[\frac{7k-8}{4}]}t^k\ :\ \text{each } r_j \text{ is a discrete function}\right\},$$

We will prove that for $\alpha > 0$:

$$L_{2\alpha} \in 7^{\alpha} X_A, \tag{3.29}$$

where for a set X and a number k

$$kX := \{kx : x \in X\}.$$

Firstly from Appendix A we see that that in each case there is an integer l and discrete functions $a_{k,u}^{(i)}(n)$ and $b_{k,u}^{(i)}(n)$ for $l \le k \le l+6$ such that

$$\begin{aligned} U_A(ut^k) = &\sum_{n \ge [(k+7)/7]} a_{k,u}^{(0)}(n) 7^{[\frac{7n-k-5}{4}]} t^n + p_0 \sum_{n \ge [(k+7)/7]} a_{k,u}^{(1)}(n) 7^{[\frac{7n-k-5}{4}]} t^n \\ &+ \sum_{n \ge [(k+14)/7]} a_{k,u}^{(2)}(n) 7^{[\frac{7n-k-8}{4}]} t^n, \end{aligned} \tag{3.30}$$

$$\begin{aligned} U_B(ut^k) = &\sum_{n \ge [k/7]} b_{k,u}^{(0)}(n) 7^{[\frac{7n-k+5}{4}]} t^n + p_0 \sum_{n \ge [k/7]} b_{k,u}^{(1)}(n) 7^{[\frac{7n-k+5}{4}]} t^n \\ &+ \sum_{n \ge [(k+6)/7]} b_{k,u}^{(2)}(n) 7^{[\frac{7n-k+2}{4}]} t^n, \end{aligned} \tag{3.31}$$

where u is one of 1, p_0 or p_1. Then using Lemma 3.3 and Lemma 3.4, we find that (3.30) and (3.31) hold for all $k \in \mathbb{N}$. Next, we prove (3.29) inductively by proving the following three statements:

$$\begin{aligned} &L_1 \in X_B, \\ &g \in X_B \text{ implies } U_B(g) \in 7X_A, \quad \text{and} \\ &g \in X_A \text{ implies } U_A(g) \in X_B. \end{aligned}$$

Let $k = 0$ in (3.30) we can see that

$$\begin{aligned} L_1 =& U_A(L_0) = 7U_A(p_0) + U_A(p_1) \\ =& \sum_{n=1}^{\infty} r_1(n) 7^{[\frac{7n-5}{4}]} t^n + p_0 \sum_{n=1}^{\infty} r_2(n) 7^{[\frac{7n-5}{4}]} t^n + \sum_{n=2}^{\infty} r_3(n) 7^{[\frac{7n-8}{4}]} t^n \in X_B, \end{aligned}$$

with some discrete functions r_i. Assume that $g \in X_B$. There are discrete functions r_i such that

$$g = \sum_{k=1}^{\infty} r_1(k) 7^{[\frac{7k-5}{4}]} t^k + p_0 \sum_{k=1}^{\infty} r_2(k) 7^{[\frac{7k-5}{4}]} t^k + p_1 \sum_{k=2}^{\infty} r_3(k) 7^{[\frac{7k-8}{4}]} t^k.$$

This implies that

$$U_B(g) = \sum_{k=1}^{\infty} r_1(k) 7^{[\frac{7k-5}{4}]} U_B(t^k) + \sum_{k=1}^{\infty} r_2(k) 7^{[\frac{7k-5}{4}]} U_B(p_0 t^k) + \sum_{k=2}^{\infty} r_3(k) 7^{[\frac{7k-8}{4}]} U_B(p_1 t^k). \tag{3.32}$$

Each sum in (3.32) can be written in the form $7g_1$ for some $g_1 \in X_A$. Since the proofs are similar we only consider the first sum. From (3.31)

$$\sum_{k=1}^{\infty} r_1(k) 7^{[\frac{7k-5}{4}]} U_B(t^k) = \sum_{k=1}^{\infty}\sum_{n=0}^{\infty} r_1(k)(b_{k,1}^{(0)}(n) + b_{k,1}^{(1)}(n)) 7^{[\frac{7k-5}{4}]+[\frac{7n-k+5}{4}]} t^n + \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} r_1(k) b_{k,1}^{(2)}(n) 7^{[\frac{7k-5}{4}]+[\frac{7n-k+2}{4}]} t^n. \tag{3.33}$$

Observe that for $k = 1$:

$$\left[\frac{7k-5}{4}\right] + \left[\frac{7n-k+5}{4}\right] = \left[\frac{7n+4}{4}\right] = \left[\frac{7n}{4}\right] + 1,$$
$$\left[\frac{7k-5}{4}\right] + \left[\frac{7n-k+2}{4}\right] = \left[\frac{7n+1}{4}\right] = \left[\frac{7n-3}{4}\right] + 1,$$

and for $k > 1$:

$$\left[\frac{7k-5}{4}\right] + \left[\frac{7n-k+5}{4}\right] \geq \left[\frac{7n+6k-3}{4}\right] \geq \left[\frac{7n+9}{4}\right] \geq \left[\frac{7n}{4}\right] + 1,$$
$$\left[\frac{7k-5}{4}\right] + \left[\frac{7n-k+2}{4}\right] \geq \left[\frac{7n+6k-6}{4}\right] \geq \left[\frac{7n+6}{4}\right] \geq \left[\frac{7n-3}{4}\right] + 1.$$

Hence the right hand side of (3.32) can be written in the form $7g_1$ for some $g_1 \in X_A$. The statement that $g \in X_A$ implies $U_A(g) \in X_B$ can be proved analogously. So that we have proved (3.29) which implies that

$$A(7^{2\alpha}n + \lambda_{2\alpha}) \equiv 0 \pmod{7^\alpha}, \tag{3.34}$$

and noting that $7^{2\alpha+1}n + \lambda_{2\alpha+1}$ is a subsequence of $7^{2\alpha}n + \lambda_{2\alpha}$, we have

$$A(7^{2\alpha+1}n + \lambda_{2\alpha+1}) \equiv 0 \pmod{7^\alpha}. \tag{3.35}$$

Congruences (3.34) and (3.35) are each cases of (1.7) after replacing 2α and $2\alpha + 1$ by α, where noting that for $\alpha > 0$:

$$\begin{aligned} A(7^{2\alpha}n + \lambda_{2\alpha}) &= a_f(7^{2\alpha-1}n + \lambda_{2\alpha}/7) - a_f(7^{2\alpha+1}n + 7\lambda_{2\alpha} - 2) \\ &= a_f(7^{2\alpha-1}n + \delta_{2\alpha-1}) - a_f(7^{2\alpha+1}n + \delta_{2\alpha+1}), \end{aligned}$$

and

$$\begin{aligned} A(7^{2\alpha+1}n + \lambda_{2\alpha+1}) &= a_f(7^{2\alpha}n + \lambda_{2\alpha+1}/7) - a_f(7^{2\alpha+2}n + 7\lambda_{2\alpha+1} - 2) \\ &= a_f(7^{2\alpha}n + \delta_{2\alpha}) - a_f(7^{2\alpha+2}n + \delta_{2\alpha+2}). \end{aligned}$$

This completes the proof of Theorem 1.4.

4. The crank parity function modulo powers of 7

4.1. Preliminary Lemmas. We denote

$$A := \frac{\eta(\tau)^3\eta(98\tau)^2}{\eta(2\tau)^2\eta(49\tau)^3} = \frac{q^2J_1^3J_{98}^2}{J_2^2J_{49}^3},$$

and for any $g:\mathbb{H}\to\mathbb{C}$, we define

$$U^{(1)}(g) := U_7(Ag),\ U^{(0)}(g) := U_7(g),$$

where $U_p(f)$ is defined by (2.20). Let $L_0 := 1$ and for $\alpha \geq 0$,

$$L_{2\alpha+1} := U^{(1)}(L_{2\alpha}),\ L_{2\alpha+2} := U^{(0)}(L_{2\alpha+1}).$$

Using (2.19) and (2.21), it is easy to verify that

$$L_{2\alpha} = \frac{J_2^2}{J_1^3}\sum_{n=0}^{\infty}\beta(7^{2\alpha}n+\delta_{2\alpha})q^n, \quad \text{for } \alpha\geq 1 \text{ and}$$

$$L_{2\alpha+1} = \frac{J_{14}^2}{J_7^3}\sum_{n=0}^{\infty}\beta(7^{2\alpha+1}n+\delta_{2\alpha+1})q^n,$$

for $\alpha \geq 0$. Since $\frac{J_2^2}{J_1^3}$ and $\frac{J_{14}^2}{J_7^3}$ have leading coefficient 1, the congruence (1.5) is equivalent to

$$L_{\alpha+2} \equiv K_\alpha L_\alpha \pmod{7^\alpha}. \tag{4.1}$$

In order to prove (4.1), we use the forty-two fundamental relations in Appendix B. Again these identities can be proved using the algorithm described in [14, Section 2C, pp.8-9]. We note $t = \eta(7\tau)^4/\eta(\tau)^4$ and

$$p_0 := \frac{\eta(14\tau)^4\eta(\tau)^4}{\eta(7\tau)^4\eta(2\tau)^4} = \frac{qJ_{14}^4J_1^4}{J_7^4J_2^4},$$

$$p_1 := \frac{1}{7}\left(\frac{\eta(14\tau)\eta(\tau)^7}{\eta(7\tau)\eta(2\tau)^7} - 1\right) = \frac{1}{7}\left(\frac{J_{14}J_1^7}{J_7J_2^7} - 1\right).$$

It is clear the q-expansion of p_0 has integer coefficients. Let $u(q) := \frac{J_1}{J_2}$. Since $u(q^7) \equiv u(q)^7 \pmod 7$, the q-expansion of $p_1 = \frac{1}{7}(\frac{u(q)^7}{u(q^7)} - 1)$ also has integer coefficients. To prove Theorem 1.2, some lemmas are needed.

Lemma 4.1. *For $j = 0,\ 1,\ 2$ there exist discrete functions of n, $a_{k,i}(n,j)$ and $b_{k,i}(n,j)$ such that*

$$\begin{aligned}U^{(1)}(p_0t^k) =& p_0\sum_{n\geq[k/7]}a_{k,0}(n,0)7^{[\frac{7n-k+1}{4}]}t^n + p_1\sum_{n\geq[k/7]}a_{k,0}(n,1)7^{[\frac{7n-k+5}{4}]}t^n\\ &+\sum_{n\geq[(k+7)/7]}a_{k,0}(n,2)7^{[\frac{7n-k+1}{4}]}t^n,\end{aligned} \tag{4.2}$$

$$U^{(1)}(p_1t^k) = p_0 \sum_{n\geq[k/7]} a_{k,1}(n,0)7^{[\frac{7n-k+1}{4}]}t^n + p_1 \sum_{n\geq[k/7]} a_{k,1}(n,1)7^{[\frac{7n-k+4}{4}]}t^n + \sum_{n\geq[(k+7)/7]} a_{k,1}(n,2)7^{[\frac{7n-k+1}{4}]}t^n, \tag{4.3}$$

$$U^{(1)}(t^k) = p_0 \sum_{n\geq[k/7]} a_{k,2}(n,0)7^{[\frac{7n-k+1}{4}]}t^n + p_1 \sum_{n\geq[k/7]} a_{k,2}(n,1)7^{[\frac{7n-k+5}{4}]}t^n + \sum_{n\geq[(k+7)/7]} a_{k,2}(n,2)7^{[\frac{7n-k+1}{4}]}t^n, \tag{4.4}$$

$$U^{(0)}(p_0t^k) = p_0 \sum_{n\geq[(k+7)/7]} b_{k,0}(n,0)7^{[\frac{7n-k-1}{4}]}t^n + p_1 \sum_{n\geq[k/7]} b_{k,0}(n,1)7^{[\frac{7n-k+2}{4}]}t^n + \sum_{n\geq[(k+7)/7]} b_{k,0}(n,2)7^{[\frac{7n-k-1}{4}]}t^n, \tag{4.5}$$

$$U^{(0)}(p_1t^k) = p_0 \sum_{n\geq[(k+7)/7]} b_{k,1}(n,0)7^{[\frac{7n-k-1}{4}]}t^n + p_1 \sum_{n\geq[k/7]} b_{k,1}(n,1)7^{[\frac{7n-k+2}{4}]}t^n + \sum_{n\geq[(k+7)/7]} b_{k,1}(n,2)7^{[\frac{7n-k-1}{4}]}t^n, \tag{4.6}$$

$$U^{(0)}(t^k) = \sum_{n\geq[(k+6)/7]} b_{k,2}(n,2)7^{[\frac{7n-k-1}{4}]}t^n. \tag{4.7}$$

Proof. Firstly from Appendix A we see that that in each case there is an integer l and discrete functions $a^{(i)}_{k,u}(n)$ and $b^{(i)}_{k,u}(n)$ for $l \leq k \leq l+6$ such that From Appendix B we see that in each of (4.2)-(4.7) there is an integer l and appropriate discrete functions for $l \leq k \leq l+6$. Since each sum in (4.2)-(4.7) is finite, by Lemma 3.3 and Lemma 3.4, we have that (4.2)-(4.7) hold for all $k \in \mathbb{N}$. ☐

Lemma 4.2. *For each α and $i = 0,\ 1,\ 2$ there exist unique polynomials $P^{(\alpha)}_i(t)$ with integer coefficients, such that*

$$L_\alpha = p_0P^{(\alpha)}_0(t) + p_1P^{(\alpha)}_1(t) + P^{(\alpha)}_2(t). \tag{4.8}$$

Proof. From $L_0 = 1$ and (4.2)-(4.7), the existence of (4.8) is obvious. We can check that t, p_0 and $7p_1 + 1$ are modular functions on $\Gamma_0(14)$ by Theorem 2.1. Using Theorem 2.2, Theorem 2.3 and Lemma 2.4 we can calculate the order of t, p_0 and p_1 at the cusps 0 and 1/2.

$$\mathrm{Ord}(t,0,\Gamma_0(14)) = -2, \quad \mathrm{Ord}(t,1/2,\Gamma_0(14)) = -1,$$
$$\mathrm{Ord}(p_0,0,\Gamma_0(14)) = 1, \quad \mathrm{Ord}(p_0,1/2,\Gamma_0(14)) = -1,$$
$$\mathrm{Ord}(p_1,0,\Gamma_0(14)) = 0, \quad \mathrm{Ord}(p_1,1/2,\Gamma_0(14)) = -2.$$

Suppose that

$$p_0\sum_{n=0}^{M}a_nt^n+p_1\sum_{n=0}^{M}b_nt^n+\sum_{n=0}^{N}c_nt^n=0, \tag{4.9}$$

and $c_N\neq 0$. We see that the order of

$$\sum_{n=0}^{N}c_nt^n,$$

at 0 is $-2N$, and so is the order of

$$p_0\sum_{n=0}^{M}a_nt^n+p_1\sum_{n=0}^{M}b_nt^n,$$

at 0. Each term p_0t^n has order $1-2n$ and each term p_1t^m has the different order $-2m$ at 0. This implies $b_N\neq 0$ and for $n,m>N$ we have $a_n=b_m=0$. The order at 1/2 of

$$p_0\sum_{n=0}^{N}a_nt^n+p_1\sum_{n=0}^{N}b_nt^n,$$

is $-2-N$ but for

$$\sum_{n=0}^{N}c_nt^n,$$

it is $-N$. This is a contradiction. It implies that there is no n such that $c_n\neq 0$ and (4.9) holds. This means that

$$p_0\sum_{n=0}^{M}a_nt^n+p_1\sum_{n=0}^{M}b_nt^n=0.$$

But the order of each term at 0 is different. We have $a_n=b_n=0$ for all n, which means for each $N>0$ the functions $1,t,...,t^N,p_0,p_0t,...,p_0t^N,p_1,p_1t,...,p_1t^N$ are linear independent. Hence the expression in (4.8) is unique. □

We need lower-bounds for the 7-adic order of coefficients.

Lemma 4.3. *For $\alpha\geq 1$ there exist integers $d_{n,i}^{(\alpha)}$, $i=0,1,2$ such that*

$$L_{2\alpha-1}=p_0\sum_{k=0}^{\infty}d_{k,0}^{(2\alpha-1)}7^{[\frac{7k+1}{4}]}t^k+p_1\sum_{k=0}^{\infty}d_{k,1}^{(2\alpha-1)}7^{[\frac{7k+4}{4}]}t^k+\sum_{k=1}^{\infty}d_{k,2}^{(2\alpha-1)}7^{[\frac{7k+1}{4}]}t^k, \tag{4.10}$$

and

$$L_{2\alpha}=p_0\sum_{k=1}^{\infty}d_{k,0}^{(2\alpha)}7^{[\frac{7k-1}{4}]}t^k+p_1\sum_{k=0}^{\infty}d_{k,1}^{(2\alpha)}7^{[\frac{7k+2}{4}]}t^k+\sum_{k=1}^{\infty}d_{k,2}^{(2\alpha)}7^{[\frac{7k-1}{4}]}t^k. \tag{4.11}$$

Proof. From Appendix B we see that $L_1 = 2p_0 + 7p_1$ has form given in (4.10). Assume that $L_{2\alpha-1}$ has the for given in (4.10) for a fixed α, then

$$
\begin{aligned}
L_{2\alpha} =& U^{(0)}(L_{2\alpha-1}) \\
=& \sum_{k=0}^{\infty} d_{k,0}^{(2\alpha-1)} 7^{[\frac{7k+1}{4}]} U^{(0)}(p_0 t^k) + \sum_{k=0}^{\infty} d_{k,1}^{(2\alpha-1)} 7^{[\frac{7k+4}{4}]} U^{(0)}(p_1 t^k) \\
&+ \sum_{k=1}^{\infty} d_{k,2}^{(2\alpha-1)} 7^{[\frac{7k+1}{4}]} U^{(0)}(t^k).
\end{aligned}
\tag{4.12}
$$

We wish to show that the form of each sum on the right side of (4.12) satisfies the form in (4.11). Since the proofs are similar we just consider the first sum. From (4.5),

$$
\begin{aligned}
&\sum_{k=0}^{\infty} d_{k,0}^{(2\alpha-1)} 7^{[\frac{7k+1}{4}]} U^{(0)}(p_0 t^k) \\
=& p_0 \sum_{k=0}^{\infty} \sum_{n \geq [(k+7)/7]} b_0(n,k;p_0) d_{k,0}^{(2\alpha-1)} 7^{[\frac{7n-k-1}{4}]+[\frac{7k+1}{4}]} t^n \\
&+ p_1 \sum_{k=0}^{\infty} \sum_{n \geq [k/7]} b_1(n,k;p_0) d_{k,0}^{(2\alpha-1)} 7^{[\frac{7n-k+2}{4}]+[\frac{7k+1}{4}]} t^n \\
&+ \sum_{k=0}^{\infty} \sum_{n \geq [(k+7)/7]} b_2(n,k;p_0) d_{k,0}^{(2\alpha-1)} 7^{[\frac{7n-k-1}{4}]+[\frac{7k+1}{4}]} t^n.
\end{aligned}
$$

For $k = 0$, we have

$$
\left[\frac{7n-k-1}{4}\right] + \left[\frac{7k+1}{4}\right] = \left[\frac{7n-1}{4}\right],
$$

$$
\left[\frac{7n-k+2}{4}\right] + \left[\frac{7k+1}{4}\right] = \left[\frac{7n+2}{4}\right],
$$

and for $k \geq 1$, we have

$$
\left[\frac{7n-k-1}{4}\right] + \left[\frac{7k+1}{4}\right] \geq \left[\frac{7n-k-1+7k+1-3}{4}\right] \geq \left[\frac{7n-1}{4}\right],
$$

$$
\left[\frac{7n-k+2}{4}\right] + \left[\frac{7k+1}{4}\right] \geq \left[\frac{7n-k+2+7k+1-3}{4}\right] \geq \left[\frac{7n+2}{4}\right].
$$

So that the first sum in the right side of (4.12) has the form of (4.11). Similarly, also the second and third sums have the correct form. Hence $L_{2\alpha}$ has the desired form. The proof that the correct form of $L_{2\alpha}$ implies the correct form of $L_{2\alpha+1}$ is analogous. The general result follows by induction. □

By Lemmas 4.2 and 4.3 we let

$$P_i^{(\alpha)}(t) := \sum_{n=0}^{\infty} l_{n,i}^{(\alpha)} t^n,$$

for $i = 0, 1, 2$ be the unique polynomials such that

$$L_\alpha = p_0 \sum_{n=0}^{\infty} l_{n,0}^{(\alpha)} t^n + p_1 \sum_{n=0}^{\infty} l_{n,1}^{(\alpha)} t^n + \sum_{n=0}^{\infty} l_{n,2}^{(\alpha)} t^n.$$

Next we define

$$D^{(\alpha)}(l_{m,i}, l_{n,j}) := l_{m,i}^{(\alpha)} l_{n,j}^{(\alpha+2)} - l_{m,i}^{(\alpha+2)} l_{n,j}^{(\alpha)},$$

and denote $\pi(n)$ be the 7-adic order of n (i.e. the highest power of 7 that divides n).

Lemma 4.4. *For $\alpha \geq 1$, $i, j = 0, 1, 2$,*

$$\pi(D^{(2\alpha-1)}(l_{m,i}, l_{n,j})) \geq 2\alpha - 2 + m + n + \max(\lambda_i, \lambda_j), \tag{4.13}$$

$$\pi(D^{(2\alpha)}(l_{m,i}, l_{n,j})) \geq 2\alpha - 1 + m + n + \lambda_i \lambda_j, \tag{4.14}$$

where $\lambda_0 = \lambda_2 = 0$ and $\lambda_1 = 1$.

Proof. Since $l_{n,i}^{(1)} = 0$ except $l_{0,0}^{(1)} = 2$ and $l_{0,1}^{(1)} = 7$, from (4.10) we have

$$\pi(D^{(1)}(l_{m,i}, l_{n,j})) = \infty,$$

when $m, n > 0$ and

$$\begin{aligned}\pi(D^{(1)}(l_{0,i}, l_{n,j})) =& \pi(-D^{(1)}(l_{n,j}, l_{0,i})) = \pi(l_{0,i}^{(1)} l_{n,j}^{(3)}) \\ \geq& \lambda_i + \left[\frac{7n + \mu_j}{4}\right] = \lambda_i + n + \left[\frac{3n + \mu_j}{4}\right] \geq n + \max(\lambda_i, \lambda_j),\end{aligned}$$

when $n > 0$ where $\mu_0 = \mu_2 = 1$ and $\mu_1 = 4$ and

$$\pi(D^{(1)}(l_{0,0}, l_{0,1})) = \pi(l_{0,0}^{(1)} l_{0,1}^{(3)} - l_{0,0}^{(3)} l_{0,1}^{(1)}) \geq \min(l_{0,1}^{(1)}, l_{0,1}^{(3)}) \geq 1 = \max(\lambda_0, \lambda_1),$$

which proves (4.13) for $\alpha = 1$. Suppose now that (4.13) holds for fixed α. We will compare the coefficients on both sides of

$$\begin{aligned}&p_0 \sum_{n=0}^{\infty} l_{n,0}^{(2\alpha)} t^n + p_1 \sum_{n=0}^{\infty} l_{n,1}^{(2\alpha)} t^n + \sum_{n=0}^{\infty} l_{n,2}^{(2\alpha)} t^n \\ &= \sum_{k=0}^{\infty} l_{k,0}^{(2\alpha-1)} U^{(0)}(p_0 t^k) + \sum_{k=0}^{\infty} l_{k,1}^{(2\alpha-1)} U^{(0)}(p_1 t^k) + \sum_{k=0}^{\infty} l_{k,2}^{(2\alpha-1)} U^{(0)}(t^k).\end{aligned}$$

For convenience, denote $p_2 := 1$ and for $u = 0, 1, 2$ let

$$U^{(0)}(p_u t^k) = p_0 \sum_n x_{k,u}(n, 0) t^n + p_1 \sum_n x_{k,u}(n, 1) t^n + \sum_n x_{k,u}(n, 2) t^n.$$

We have

$$l_{n,i}^{(2\alpha)} = \sum_{k,u} x_{k,u}(n,i) l_{k,u}^{(2\alpha-1)},$$

which shows that

$$D^{(2\alpha)}(l_{m,i}, l_{n,j}) = \sum_{k,r,u,v} x_{k,u}(m,i) x_{r,v}(n,j) D^{(2\alpha-1)}(l_{k,u}, l_{r,v}). \tag{4.15}$$

From (4.5)-(4.7), we have $\pi(x_{k,u}(m,i)) \geq [\frac{7m-k+\nu_i}{4}]$, where $\nu_0 = \nu_2 = -1$ and $\nu_1 = 2$. So that

$$\begin{aligned} &\pi(D^{(2\alpha)}(l_{m,i}, l_{n,j})) \\ =&\pi\left(\sum_{k,r,u,v} x_{k,u}(m,i) x_{r,v}(n,j) D^{(2\alpha-1)}(l_{k,u}, l_{r,v})\right) \\ \geq& \min_{(k,u)\neq(r,v)} \left(k + r + 2\alpha - 2 + \max(\lambda_u, \lambda_v) + \left[\frac{7m-k+\nu_i}{4}\right] + \left[\frac{7n-r+\nu_j}{4}\right]\right) \\ =& \min_{(k,u)\neq(r,v)} \left(m + n + 2\alpha - 2 + \max(\lambda_u, \lambda_v) + \left[\frac{3m+3k+\nu_i}{4}\right] + \left[\frac{3n+3r+\nu_j}{4}\right]\right). \end{aligned} \tag{4.16}$$

Noting (4.11), $l_{0,i}^{(2\alpha)} \neq 0$ only if $i = 1$, which is $3m + \nu_i > 1$ and also $3n + \nu_j > 1$. Further more, if $i = j = 1$ then $m > 0$ or $n > 0$ which is $3m + \nu_i \geq 5$ or $3n + \nu_j \geq 5$. Since $i = j = 1$ if and only if $\lambda_i \lambda_j = 1$,

$$\left[\frac{3m+3k+\nu_i}{4}\right] + \left[\frac{3n+3r+\nu_j}{4}\right] \geq \left[\frac{3k+1}{4}\right] + \left[\frac{3r+1}{4}\right] + \lambda_i \lambda_j. \tag{4.17}$$

Also noting (4.10), $l_{0,2}^{(2\alpha-1)} = 0$. This implies that if $k = r = 0$, then one of u or v is 1, which means one of λ_u or λ_v is 1. So that

$$\max(\lambda_u, \lambda_v) + \left[\frac{3k+1}{4}\right] + \left[\frac{3r+1}{4}\right] \geq 1. \tag{4.18}$$

From (4.16)-(4.18)

$$\pi(D^{(2\alpha)}(l_{m,i}, l_{n,j})) \geq 2\alpha - 1 + m + n + \lambda_i \lambda_j,$$

which is (4.14) hold for α. Then suppose that (4.14) holds for α. Let

$$U^{(1)}(p_u t^k) = p_0 \sum_n y_{k,u}(n,0) t^n + p_1 \sum_n y_{k,u}(n,1) t^n + \sum_n y_{k,u}(n,2) t^n.$$

Similar to (4.15), we have

$$D^{(2\alpha+1)}(l_{m,i}, l_{n,j}) = \sum_{k,r,u,v} y_{k,u}(m,i) y_{r,v}(n,j) D^{(2\alpha)}(l_{k,u}, l_{r,v}).$$

From (4.2)-(4.4), we have $\pi(y_{k,u}(m,i)) \geq [\frac{7m-k+\mu_{i,u}}{4}]$, where $\mu_{0,u} = \mu_{2,u} = 1$ and $\mu_{1,u} = 5-\lambda_u$. So that

$$\begin{aligned}
(4.19)\qquad & \pi(D^{(2\alpha+1)}(l_{m,i}, l_{n,j})) \\
=& \pi\left(\sum_{k,r,u,v} y_{k,u}(m,i) y_{r,v}(n,j) D^{(2\alpha)}(l_{k,u}, l_{r,v})\right) \\
\geq & \min_{(k,u)\neq(r,v)} \left(k + r + 2\alpha - 1 + \lambda_u\lambda_v + \left[\frac{7m-k+\mu_{i,u}}{4}\right] + \left[\frac{7n-r+\mu_{j,v}}{4}\right]\right) \\
=& \min_{(k,u)\neq(r,v)} \left(m + n + 2\alpha - 1 + \lambda_u\lambda_v + \left[\frac{3m+3k+\mu_{i,u}}{4}\right] + \left[\frac{3n+3r+\mu_{j,v}}{4}\right]\right).
\end{aligned}$$

Also from (4.11) and then $l_{0,u}^{(2\alpha)} \neq 0$ only if $u = 1$, we can see that $k \geq 1$ or $r \geq 1$ which is $3k + \mu_{i,u} \geq 4$ or $3r + \mu_{j,v} \geq 4$. When $u = v = 1$,

$$\lambda_u\lambda_v + \left[\frac{3m+3k+\mu_{i,u}}{4}\right] + \left[\frac{3n+3r+\mu_{j,v}}{4}\right] \geq 2 \geq \max(\lambda_i, \lambda_j) + 1. \tag{4.20}$$

When $u \neq 1$,($v \neq 1$ are analogous), then $k \geq 1$ and

$$\lambda_u\lambda_v + \left[\frac{3m+3k+\mu_{i,u}}{4}\right] + \left[\frac{3n+3r+\mu_{j,v}}{4}\right] \geq \left[\frac{3m+3+\mu_{i,u}}{4}\right] + \left[\frac{3n+\mu_{j,v}}{4}\right], \tag{4.21}$$

if $j = 1$, then $\mu_{j,v} \geq 4$ and

$$\left[\frac{3m+3+\mu_{i,u}}{4}\right] + \left[\frac{3n+\mu_{j,v}}{4}\right] \geq 2 \geq \max(\lambda_i, \lambda_j) + 1, \tag{4.22}$$

if $i = 1$, then $\mu_{i,u} \geq 5$ and

$$\left[\frac{3m+3+\mu_{i,u}}{4}\right] + \left[\frac{3n+\mu_{j,v}}{4}\right] \geq 2 \geq \max(\lambda_i, \lambda_j) + 1, \tag{4.23}$$

if $i, j \neq 1$,

$$\left[\frac{3m+3+\mu_{i,u}}{4}\right] + \left[\frac{3n+\mu_{j,v}}{4}\right] \geq 1 \geq \max(\lambda_i, \lambda_j) + 1. \tag{4.24}$$

From (4.22)-(4.24) we have (4.21), and from (4.19)-(4.21) we have

$$\pi(D^{(2\alpha+1)}(l_{m,i}, l_{n,j})) \geq 2\alpha + m + n + \max(\lambda_i, \lambda_j),$$

which is (4.13) hold for $\alpha + 1$. Hence (4.13) and (4.14) hold for all $\alpha \geq 1$. $\square$

4.2. **Prove of Theorem 1.2.**

Proof. From Lemma 4.4, for each $(n,j)\neq(0,0)$, by $l_{0,2}^{(2\alpha-1)}=0$ we have

$$\pi(D^{(2\alpha-1)}(l_{0,0},l_{n,j}))\geq 2\alpha-1, \tag{4.25}$$

and for $(n,j)\neq(0,1)$, by $l_{0,0}^{(2\alpha)}=0$ and $l_{0,2}^{(2\alpha)}=0$ we have

$$\pi(D^{(2\alpha)}(l_{0,1},l_{n,j}))\geq 2\alpha. \tag{4.26}$$

It is easy to see that for $\alpha\geq 1$,

$$L_{2\alpha-1}\equiv l_{0,0}^{(2\alpha-1)}p_0 \pmod 7,$$

and

$$L_{2\alpha}\equiv l_{0,1}^{(2\alpha)}p_1 \pmod 7.$$

Then we notice that $7\nmid l_{0,0}^{(2\alpha-1)}$ and $7\nmid l_{0,1}^{(2\alpha)}$ which can be implied from $l_{0,0}^{(1)}=2$, $U^{(0)}(p_0)\equiv 3p_1 \pmod 7$ and $U^{(1)}(p_1)\equiv 2p_0 \pmod 7$. Let $x_{2\alpha-1}$ be a solution of

$$l_{0,0}^{(2\alpha-1)}\equiv x l_{0,0}^{(2\alpha+1)} \pmod{7^{2\alpha-1}},$$

then for $(n,i)\neq(0,0)$, from (4.25),

$$l_{n,i}^{(2\alpha-1)}l_{0,0}^{(2\alpha-1)}\equiv x_{2\alpha-1}l_{0,0}^{(2\alpha+1)}l_{n,i}^{(2\alpha-1)}\equiv x_{2\alpha-1}l_{0,0}^{(2\alpha-1)}l_{n,i}^{(2\alpha+1)} \pmod{7^{2\alpha-1}}.$$

Cancelling $l_{0,0}^{(2\alpha-1)}$ we obtain

$$l_{n,i}^{(2\alpha-1)}\equiv x_{2\alpha-1}l_{n,i}^{(2\alpha+1)} \pmod{7^{2\alpha-1}}, \tag{4.27}$$

Similarly, let $x_{2\alpha}$ be a solution of

$$l_{0,1}^{(2\alpha)}\equiv x l_{0,1}^{(2\alpha)} \pmod{7^{2\alpha}},$$

then for $(n,i)\neq(0,1)$, from (4.26),

$$l_{n,i}^{(2\alpha)}l_{0,1}^{(2\alpha)}\equiv x_{2\alpha}l_{0,1}^{(2\alpha+2)}l_{n,i}^{(2\alpha)}\equiv x_{2\alpha}l_{0,1}^{(2\alpha)}l_{n,i}^{(2\alpha+2)} \pmod{7^{2\alpha}}.$$

Cancelling $l_{0,1}^{(2\alpha)}$ we obtain

$$l_{n,i}^{(2\alpha)}\equiv x_{2\alpha}l_{n,i}^{(2\alpha+2)} \pmod{7^{2\alpha}}. \tag{4.28}$$

From (4.8), (4.27) and (4.28) prove (4.1) and also (1.5). □

5. The congruence modulo powers of 5 and 7 for $\omega(q)$

5.1. The U_P^* operator. Let p prime and f be a function defined on the upper-half plane (not necessarily a modular function). We define

$$f \;\Big|\; U_p^* := \frac{1}{p}\sum_{j=0}^{p-1} f \;\Bigg|\; \begin{pmatrix} 1/p & 24j/p \\ 0 & 1 \end{pmatrix} = \frac{1}{p}\sum_{j=0}^{p-1} f\left(\frac{\tau+24j}{p}\right),$$

and we note that for a function $g(\tau)$

$$g \;\Big|\; U_p^* = g^* \;|\, U_p\;(\tau/24),$$

where $g^*(\tau) = g(24\tau)$.

5.2. Modular properties. Consider the following third order mock theta functions:

$$f(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2},$$

$$\omega(q) := \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q;q^2)_n^2}.$$

Following [32], we define $F := (f_0, f_1, f_2)^T$ by:

$$f_0(\tau) := q^{-1/24} f(q),$$
$$f_1(\tau) := 2q^{1/3}\omega(q^{\frac{1}{2}}),$$
$$f_2(\tau) := 2q^{1/3}\omega(q^{-\frac{1}{2}}).$$

Also define

$$G(\tau) := 2i\sqrt{3}\int_{-\overline{\tau}}^{i\infty} \frac{(g_1(z), g_0(z), -g_2(z))^T}{\sqrt{-i(z+\tau)}} dz,$$

where

$$g_0(z) := \sum_{n\in\mathbb{Z}} (-1)^n (n+1/3) e^{3\pi i (n+1/3)^2 z},$$

$$g_1(z) := -\sum_{n\in\mathbb{Z}} (n+1/6) e^{3\pi i (n+1/6)^2 z},$$

$$g_2(z) := \sum_{n\in\mathbb{Z}} (n+1/3) e^{3\pi i (n+1/3)^2 z}.$$

Letting $H = (h_0, h_1, h_2)^T := F - G$, [32, Theorem 3.6] gives

$$H(\tau+1) = \begin{pmatrix} \zeta_{24}^{-1} & 0 & 0 \\ 0 & 0 & \zeta_3 \\ 0 & \zeta_3 & 0 \end{pmatrix} H(\tau), \tag{5.1}$$

where $\zeta_m := e^{2\pi i/m}$, and

$$\frac{1}{\sqrt{-i\tau}} H(-1/\tau) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} H(\tau). \tag{5.2}$$

Lemma 5.1. *For each prime $p \geq 5$,*

$$G \mid U_p^* = \chi_6(p) G(p\tau),$$

where

$$\chi_6(p) = \begin{cases} 1 & p \equiv 1 \pmod 6, \\ -1 & p \equiv -1 \pmod 6, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. We prove that for $i = 0, 1, 2$,

$$g_i \mid U_p^* = p\chi_6(p) g_i(p\tau). \tag{5.3}$$

Since the proofs are similar, we only prove the case $i = 0$. By definition,

$$g_0(\tau) := \sum_{n\in\mathbb{Z}} (-1)^n (n + 1/3) e^{3\pi i (n+1/3)^2 \tau}.$$

So that

$$g_0 \mid U_p^* = \frac{1}{p} \sum_{j=0}^{p-1} \sum_{n\in\mathbb{Z}} (-1)^n (n+1/3) e^{3\pi i (n+1/3)^2 \tau/p + 3\pi i (n+1/3)^2 \cdot 24j/p}. \tag{5.4}$$

Noting that $3\pi i(n+1/3)^2 \cdot 24j = 2\pi i \cdot 4j(3n+1)^2$ and $4(3n+1)^2 \equiv 0 \pmod p$ if and only if $3n+1 \equiv 0 \pmod p$, which implies

$$\begin{aligned} &\sum_{j=0}^{p-1} \sum_{\substack{n\in\mathbb{Z} \\ p\nmid 3n+1}} (-1)^n (n+1/3) e^{3\pi i (n+1/3)^2 \tau/p + 3\pi i (n+1/3)^2 \cdot 24j/p} \\ &= \sum_{\substack{n\in\mathbb{Z} \\ p\nmid 3n+1}} \sum_{j=0}^{p-1} \zeta_p^{4j(3n+1)^2} (-1)^n (n+1/3) e^{3\pi i (n+1/3)^2 \tau/p} = 0. \end{aligned} \tag{5.5}$$

Letting $3n+1 = kp$ then $k = 3m^* + \chi_6(p)$ for $m^* \in \mathbb{Z}$, and letting $m = \chi_6(p) m^*$ then $n + 1/3 = p\chi_6(p)(m+1/3)$ and $(-1)^n = (-1)^m$, by (5.4) and (5.5) we have

$$\begin{aligned} g_0 \mid U_p^* &= \sum_{3n+1\equiv 0 \pmod p} (-1)^n (n+1/3) e^{3\pi i (n+1/3)^2 \tau/p} \\ &= \sum_{m\in\mathbb{Z}} (-1)^m p\chi_6(p)(m+1/3) e^{3\pi i (m+1/3)^2 p\tau} \\ &= p\chi_6(p) g_0(p\tau). \end{aligned}$$

Noting that for $\tau = x + yi$

$$G(\tau) = 2i\sqrt{3}\int_{-\overline{\tau}}^{i\infty} \frac{(g_1(z), g_0(z), -g_2(z))^T}{\sqrt{-i(z+\tau)}}dz \tag{5.6}$$
$$= -2\sqrt{3}\int_{y}^{\infty} \frac{(g_1(-x+it), g_0(-x+it), -g_2(-x+it))^T}{\sqrt{y+t}}dt.$$

Using (5.3), (5.6) and $g_i(\tau) = g_i(\tau + 24), (i = 0, 1, 2)$ we easily to find that

$$G \;\Big|\; U_p^* = \chi_6(p)G(p\tau)$$

□

From Lemma 5.1, we have

$$F \;\Big|\; U_p^* - \chi_6(p)F(p\tau) = H \;\Big|\; U_p^* - \chi_6(p)H(p\tau). \tag{5.7}$$

This will enable us to find modular-properties of $f(q)$ and $\omega(q)$. We say a function f : $\mathbb{H} \longrightarrow \mathbb{C}$ is *on* Γ if for each $A \in \Gamma$:

$$f \mid A = f; \quad \text{i.e.} \quad f(A\tau) = f(\tau),$$

for all $A \in \Gamma$ and all $\tau \in \mathbb{H}$. We note that $H(\tau)$ is a non-holomorphic modular function. Many properties of modular functions also hold for non-holomorphic modular functions. For example, Atkin and Lehner's [5, Lemma 7] holds even if $f(\tau)$ is not holomorphic. Hence we have

Lemma 5.2. *Let p be prime. If f is on $\Gamma_0(pN)$ and $p \mid N$, then $f \mid U_p$ is on $\Gamma_0(N)$.*

If $e \parallel N$, we call the matrix

$$W_e = \begin{pmatrix} ae & b \\ cN & de \end{pmatrix}, \qquad a, b, c, d \in \mathbb{Z}, \quad det(W_e) = e$$

an Atkin-Lehner involution of $\Gamma_0(N)$.

Lemma 5.3 ([11, Corollary 2.2])**.** *Let W_e be an Atkin-Lehner involution of $\Gamma_0(N)$. Let $t > 0$ be such that $t|N$. Then*

$$\eta(tW_e\tau) = \eta\left(t\frac{ae\tau + b}{cN\tau + de}\right) = \nu_\eta(M)\left(\frac{cN\tau + de}{\delta}\right)^{1/2}\eta\left(\frac{et}{\delta^2}\tau\right),$$

where $\delta = (e, t)$, ν_η is eta-multiplier and

$$M = \begin{pmatrix} a\delta & bt/\delta \\ cN\delta/et & de/\delta \end{pmatrix} \in SL_2(\mathbb{Z}).$$

Lemma 5.4 ([12, Lemma 6])**.** *Let p be prime, $p|N$, $e \parallel N$ and $(p, e) = 1$. If $f(\tau)$ is on $\Gamma_0(N)$ then*

$$(f \mid U_p) \mid W_e = (f \mid W_e) \mid U_p\,.$$

Let

$$\widetilde{H} = (H_0, H_1, H_2) := \left(\frac{\eta(2\tau)^2}{\eta(\tau)^3} h_0(\tau), \frac{\eta(\tau/2)^2}{2\eta(\tau)^3} h_1(\tau), \frac{\eta(\tau)^3}{2\eta(\tau/2)^2\eta(\tau)^2} h_2(\tau) \right).$$

Lemma 5.5. *$H_0(\tau)$ is on $\Gamma_0(4)$.*

Proof. It is well-known that

$$\eta(\tau+1) = \zeta_{24}\eta(\tau), \tag{5.8}$$

and it is easy to calculate that

$$\eta(\tau+1/2) = \zeta_{48}\frac{\eta(2\tau)^3}{\eta(\tau)^2\eta(4\tau)^2}. \tag{5.9}$$

From (5.1), (5.8) and (5.9), we have

$$\widetilde{H}(\tau+1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix} \widetilde{H}(\tau). \tag{5.10}$$

It is also well-known that

$$\eta(-1/\tau) = \sqrt{-i\tau}\,\eta(\tau). \tag{5.11}$$

From (5.2) and (5.11), we have

$$\widetilde{H}(-1/\tau) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \widetilde{H}(\tau). \tag{5.12}$$

By [27, Proposition 4],

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 4 & -3 \end{pmatrix} \text{ and } \begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix},$$

generate $\Gamma_0(4)$. From (5.10) and (5.12) we can compute that

$$\begin{aligned}
&H_0(\tau+1) = H_0(\tau), \\
&H_0\left(\frac{\tau-1}{4\tau-3}\right) = H_1\left(-\frac{4\tau-3}{\tau-1}\right) = H_1\left(-\frac{1}{\tau-1}\right) = H_0(\tau-1) = H_0(\tau), \\
&H_0\left(\frac{3\tau-1}{4\tau-1}\right) = H_1\left(-\frac{4\tau-1}{3\tau-1}\right) = -iH_2\left(-\frac{\tau}{3\tau-1}\right) = iH_2\left(-\frac{1}{\tau}+3\right) = H_0(\tau),
\end{aligned}$$

which implies that $H_0(\tau)$ is on $\Gamma_0(4)$. $\square$

From Lemma 5.5, we can prove the following theorem.

Theorem 5.6. *For each prime $p \geq 5$,*

$$\frac{\eta(2p\tau)^2}{\eta(p\tau)^3}\left(f_0 \mid U_p^* - \chi_6(p) f_0(p\tau)\right),$$

is a weakly holomorphic modular function on $\Gamma_0(4p)$.

Proof. From Lemma 5.5

$$H_0(\tau) = \frac{\eta(2\tau)^2}{\eta(\tau)^3} h_0(\tau)$$

is on $\Gamma_0(4)$. Also by Theorem 2.1 the product

$$\frac{\eta(2p^2\tau)^2\eta(\tau)^3}{\eta(p^2\tau)^3\eta(2\tau)^2}$$

is a modular function on $\Gamma_0(2p^2)$. This implies

$$\frac{\eta(2p^2\tau)^2}{\eta(p^2\tau)^3} h_0(\tau)$$

is on $\Gamma_0(4p^2)$ and by Lemma 5.2,

$$\frac{\eta(2p^2\tau)^2}{\eta(p^2\tau)^3} h_0(\tau) \mid U_p \tag{5.13}$$

is on $\Gamma_0(4p)$. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4p),$$

then

$$A^* = \begin{pmatrix} a & bp \\ c/p & d \end{pmatrix} \in \Gamma_0(4),$$

and

$$H_0(pA\tau) = H_0(A^*(p\tau)) = H_0(p\tau). \tag{5.14}$$

By (5.7), (5.13) and (5.14), the function

$$\begin{aligned}\frac{\eta(2p\tau)^2}{\eta(p\tau)^3}\left(f_0 \mid U_p^* - \chi_6(p) f_0(p\tau)\right) =& \frac{\eta(2p\tau)^2}{\eta(p\tau)^3}\left(h_0 \mid U_p^* - \chi_6(p) h_0(p\tau)\right) \\ =& \frac{\eta(2p^2\tau)^2}{\eta(p^2\tau)^3} h_0(\tau) \mid U_p - \chi_6(p) H_0(p\tau)\end{aligned}$$

is on $\Gamma_0(4p)$, and is holomorphic on $\mathbb{H}$. By using an argument similar to that of [21, Section 5] we can show that the function satisfies condition (iii) (Section 2.1) and is thus a weakly holomorphic function on $\Gamma_0(4p)$. ∎

For example, letting $p = 5, 7$ we have

$$\frac{J_{10}^2}{J_5^3}\left(qf(q) \mid U_5 + f(q^5)\right) = \frac{J_2^4 J_{10}^4}{J_1 J_4^3 J_5^3 J_{20}} - 4q\frac{J_1^2 J_4^3 J_{10} J_{20}}{J_2^5 J_5^2}, \tag{5.15}$$

$$\frac{J_{14}^2}{J_7^3}\left(q^2 f(q) \mid U_7 - f(q^7)\right) = -\frac{J_1^3 J_7^3}{J_2^5 J_{14}} - 6q^2\frac{J_1^4 J_{14}^6}{J_2^6 J_7^4}. \tag{5.16}$$

Note that (5.15) can also be found in [14, Eq. (3.1)].

For each prime $p \geq 5$,

$$W(p) := \begin{pmatrix} p^2 - 1 & -1 \\ 4p^2 & -4 \end{pmatrix}$$

is an Atkin-Lehner involution on $\Gamma_0(4p^2)$ with $a = (p^2-1)/4$, $b = -1$, $c = 1$, $d = -1$ and $e = 4$.

Theorem 5.7. *For each prime $p \geq 5$*

(5.17)
$$\frac{\eta(2p\tau)^2}{\eta(p\tau)^3}\left(f_0(\tau) \;\middle|\; U_p^* - \chi_6(p) f_0(p\tau)\right) \mid W(p) = \frac{\eta(2p\tau)^2}{2\eta(4p\tau)^3}\left(f_1(4\tau) \;\middle|\; U_p^* - \chi_6(p) f_1(4p\tau)\right).$$

Proof. We know

$$M_p = \frac{\eta(2p^2\tau)^2 \eta(\tau)^3}{\eta(p^2\tau)^3 \eta(2\tau)^2}$$

is a modular function on $\Gamma_0(2p^2)$. From Lemma 5.5, $M_p H_0(\tau)$ is on $\Gamma_0(4p^2)$. Thus by Lemma 5.4

(5.18)
$$\frac{\eta(2p\tau)^2}{\eta(p\tau)^3}(h_0(\tau) \;|\; U_p^*) \mid W(p) = \left(M_p H_0(\tau) \;\middle|\; U_p^*\right) \mid W(p) = (M_p H_0(\tau) \mid W(p)) \;\Big|\; U_p^* .$$

Using (5.10) and (5.12), and letting $a = (p^2-1)/4$ and $\tau_1 = 4\tau$ we have

$$\begin{aligned} H_0(\tau) \mid W(p) &= H_0\left(\frac{a\tau_1 - 1}{(4a+1)\tau_1 - 4}\right) = H_1\left(-\frac{(4a+1)\tau_1 - 4}{a\tau_1 - 1}\right) \\ &= H_1\left(-\frac{\tau_1}{a\tau_1 - 1}\right) = H_0\left(a - \frac{1}{\tau_1}\right) = H_1(4\tau). \end{aligned} \tag{5.19}$$

Using Lemma 5.3 we have

$$M_p \mid W(p) = \frac{\eta(2p^2\tau)^2 \eta(4\tau)^3}{\eta(4p^2\tau)^3 \eta(2\tau)^2} \tag{5.20}$$

(5.18), (5.19) and (5.20) gives

$$\frac{\eta(2p\tau)^2}{\eta(p\tau)^3}(h_0(\tau) \;|\; U_p^*) \mid W(p) = \frac{\eta(2p\tau)^2}{2\eta(4p\tau)^3}(h_1(4\tau) \;|\; U_p^*). \tag{5.21}$$

Also, it is easy to calculate that

$$\frac{\eta(2p\tau)^2}{\eta(p\tau)^3} h_0(p\tau) \mid W(p) = H_0(p\tau) \mid W(p) = H_1(4p\tau) = \frac{\eta(2p\tau)^2}{2\eta(4p\tau)^3} h_1(4\tau). \tag{5.22}$$

(5.21) and (5.22) give

(5.23)

$$\frac{\eta(2p\tau)^2}{\eta(p\tau)^3}\left(h_0(\tau) \ \Big| \ U_p^* - \chi_6(p) h_0(p\tau)\right) \mid W(p) = \frac{\eta(2p\tau)^2}{2\eta(4p\tau)^3}\left(h_1(4\tau) \ \Big| \ U_p^* - \chi_6(p) h_1(4p\tau)\right).$$

(5.7) and (5.23) complete the proof. □

5.3. **The congruences for $\omega(q)$.** Theorem 5.7 implies that if there are congruences for the coefficients of $f(q)$, there will be congruences for the coefficients of $\omega(q)$. For example, letting $p = 5$ in (5.17), we have

$$\frac{J_{10}^2}{J_5^3}\left(q f(q) \mid U_5 + f(q^5)\right) \mid W(5) = \frac{J_{10}^2}{J_{20}^3}\left(q^{-7}\omega(q^2) \mid U_5 + q^5\omega(q^{10})\right). \tag{5.24}$$

Applying $W(5)$ to both sides of (5.15) and using (5.24) and Lemma 5.3, we obtain the generating function of $q^{-2}\omega(q^2) \mid U_5 + q^6\omega(q^{10})$ after multiplying both sides by $\frac{qJ_{20}^3}{J_{10}^2}$:

$$\sum_{n=0}^{\infty}(a_\omega(5n+1) + a_\omega((n-3)/5))q^{2n} = \frac{J_2^4 J_{10}^2}{J_1^3 J_4 J_5} + \frac{J_1^3 J_4^2 J_5 J_{20}}{J_2^5 J_{10}}. \tag{5.25}$$

Theorem 5.8. *For all $\alpha \geq 3$ and all $n \geq 0$ we have*

$$a_\omega(5^\alpha n + \delta_\alpha) + a_\omega(5^{\alpha-2} n + \delta_{\alpha-2}) \equiv 0 \pmod{5^{\lfloor \frac{1}{2}\alpha \rfloor}},$$

where δ_α satisfies $0 < \delta_\alpha < 5^\alpha$ and $3\delta_\alpha + 2 \equiv 0 \pmod{5^\alpha}$.

Proof. We define

$$a := \frac{J_{50}^2 J_4^4}{q^{12} J_{100}^4 J_2^2}, \quad b := \frac{q^4 J_{100}}{J_4},$$

and

$$P_a := q\left(\frac{J_{20} J_{10}^2 J_2^6}{J_5 J_4^5 J_1^3} + \frac{J_{20}^2 J_5 J_1^3}{J_{10} J_4^2 J_2^3}\right),$$

$$P_b := \frac{1}{q}\left(\frac{J_{10}^6 J_4 J_2^2}{J_{20}^5 J_5^3 J_1} - \frac{J_5^3 J_4^2 J_1}{J_{20}^2 J_{10}^3 J_2}\right).$$

Let $K_0 := P_a$ and

$$K_{2\alpha+1} = aK_{2\alpha} \mid U_5, \quad K_{2\alpha+2} = bK_{2\alpha+1} \mid U_5.$$

From (5.25) and a simple calculation which is similar to (4.11) and (4.10) in [14], we have

$$K_{2\alpha} = \frac{qJ_{20}J_2^2}{J_4^4}\sum_{n=0}^{\infty} a(5^{2\alpha}n+\gamma_{2\alpha})q^{2n},$$
$$K_{2\alpha+1} = \frac{J_{10}^2J_4}{qJ_{20}^4}\sum_{n=0}^{\infty} a(5^{2\alpha+1}n+\gamma_{2\alpha+1})q^{2n},$$

where $a(n) := a_\omega(5n+1)+a_\omega((n-3)/5)$, $\gamma_{2\alpha} = \frac{1}{3}(5^{2\alpha}-1)$ and $\gamma_{2\alpha+1} = \frac{1}{3}(2\cdot 5^{2\alpha+1}-1)$. Let

$$t_\omega := \frac{J_{10}^4J_4^2}{J_{20}^2J_2^4}.$$

Using Lemma 5.3, it is easy to see that

$$A \mid W(5) = a, \quad B \mid W(5) = b,$$

and

$$P_A \mid W(5) = P_a, \quad P_B \mid W(5) = P_b, \quad t \mid W(5) = t_\omega.$$

We will prove that for each $\alpha \geq 0$

(5.26) $$K_\alpha = L_\alpha \mid W(5)\ .$$

First, $K_0 = P_a = P_A \mid W(5) = L_0 \mid W(5)$, and then assume that (5.26) holds for 2α, by Lemma 5.4

$$K_{2\alpha+1} = K_{2\alpha} \mid U_5 = L_\alpha \mid W(5) \mid U_5 = L_\alpha \mid U_5 \mid W(5) = L_{2\alpha+1} \mid W(5)\ ,$$

which means (5.26) holds for $2\alpha+1$. Similarly, (5.26) holds for $2\alpha+1$ also implies that (5.26) holds for $2\alpha+2$. Inductively, (5.26) hold for each $\alpha \geq 0$. Hence

(5.27) $$K_{2\alpha} = L_\alpha \mid W(5) \in 5^\alpha X_a,$$

and

(5.28) $$K_{2\alpha+1} = L_{\alpha+1} \mid W(5) \in 5^{\alpha+1} X_b,$$

where

$$X_a := \left\{ P_a \sum_{k=1}^{\infty} r(k) 5^{[\frac{3k-3}{4}]} t_\omega^k, \quad \text{r is discrete function} \right\},$$
$$X_b := \left\{ P_b \sum_{k=2}^{\infty} r(k) 5^{[\frac{3k-6}{4}]} t_\omega^k, \quad \text{r is discrete function} \right\}.$$

(5.27) and (5.28) imply that

$$a(5^{\alpha-1}n+\gamma_{\alpha-1}) \equiv 0 \pmod{5^{\left\lfloor \frac{1}{2}\alpha \right\rfloor}},$$

so that

$$a_\omega(5^\alpha n+\delta_\alpha) + a_\omega(5^{\alpha-2}n+\delta_{\alpha-2}) \equiv 0 \pmod{5^{\left\lfloor \frac{1}{2}\alpha \right\rfloor}}.$$

This completes the proof of (1.8). The proof of (1.9) is analogous. □

Appendix A. The Fundamental Relations for the rank parity function for powers of 7

Group I

$$\begin{aligned}
U_A(1) &= 8\cdot 7^9t^6 + 176\cdot 7^7t^5 + 16\cdot 7^6t^3 + 1464\cdot 7^5t^4 + 1199\cdot 7^2t^2 + 9\cdot 7^2t \\
&\quad + p_0(7^{11}t^7 + 23\cdot 7^9t^6 + 206\cdot 7^7t^5 + 125\cdot 7^6t^4 + 242\cdot 7^4t^3 + 23\cdot 7^3t^2 \\
&\quad + 10t) + p_1(-7^{10}t^7 - 23\cdot 7^8t^6 - 198\cdot 7^6t^5 - 109\cdot 7^5t^4 - 170\cdot 7^3t^3 \\
&\quad - 9\cdot 7^2t^2),\\
U_A(t^{-1}) &= t,\\
U_A(t^{-2}) &= -6\cdot 7^2t^2 - 15\cdot 7t + 1 + 3\cdot 7p_0t + p_1(2\cdot 7^2t^2 - t),\\
U_A(t^{-3}) &= 7^4t^3 + 7^4t^2 + 15\cdot 7^2t - 7 - 3\cdot 7^2p_0t + p_1(-2\cdot 7^3t^2 + 7t),\\
U_A(t^{-4}) &= 7^6t^4 - 44\cdot 7^3t^2 - 660\cdot 7t + 44 + 132\cdot 7p_0t + p_1(88\cdot 7^2t^2 - 44t),\\
U_A(t^{-5}) &= 7^8t^5 + 5\cdot 7^5t^2 + 75\cdot 7^3t - 5\cdot 7^2 - 15\cdot 7^3p_0t + p_1(-10\cdot 7^4t^2 + 5\cdot 7^2t),\\
U_A(t^{-6}) &= 7^{10}t^6 + 12\cdot 7^9t^5 + 4\cdot 7^9t^4 + 164\cdot 7^6t^3 + 207\cdot 7^4t^2 - 2207\cdot 7^2t + 23\cdot 7^2\\
&\qquad + p_0(-2\cdot 7^9t^5 - 30\cdot 7^7t^4 - 22\cdot 7^6t^3 - 38\cdot 7^4t^2 + 477\cdot 7^2t + 2)\\
&\qquad + p_1(-2\cdot 7^{10}t^6 - 32\cdot 7^8t^5 - 26\cdot 7^7t^4 - 8\cdot 7^6t^3 + 292\cdot 7^3t^2 - 23\cdot 7^2t).
\end{aligned}$$

Group II

$$\begin{aligned}
U_A(p_0t^{-1}) &= 8\cdot 7^{11}t^6 + 176\cdot 7^9t^5 + 16\cdot 7^8t^3 + 1464\cdot 7^7t^4 + 8392\cdot 7^3t^2 + 3072\cdot 7t\\
&\quad + p_0(7^{13}t^7 + 23\cdot 7^{11}t^6 + 206\cdot 7^9t^5 + 125\cdot 7^8t^4 + 242\cdot 7^6t^3 + 23\cdot 7^5t^2\\
&\quad + 23\cdot 7^5t^2 + 512t) + p_1(-7^{12}t^7 - 23\cdot 7^{10}t^6 - 198\cdot 7^8t^5\\
&\quad - 109\cdot 7^7t^4 - 170\cdot 7^5t^3 - 3072\cdot 7t^2),\\
U_A(p_0t^{-2}) &= -6\cdot 7^3t^2 - 90\cdot 7t + 6 + p_0(7^2t^2 + 18\cdot 7t) + p_1(7^3t^3 + 13\cdot 7^2t^2 - 6t),\\
U_A(p_0t^{-3}) &= -2\cdot 7^3t^2 - 30\cdot 7t + 2 + p_0(7^4t^3 + 6\cdot 7t)\\
&\quad + p_1(7^5t^4 + 7^4t^3 + 4\cdot 7^2t^2 - 2t),\\
U_A(p_0t^{-4}) &= 22\cdot 7^3t^2 + 330\cdot 7t - 22 + p_0(7^6t^4 - 66\cdot 7t)\\
&\quad + p_1(7^7t^5 + 7^6t^4 - 44\cdot 7^2t^2 + 22t),\\
U_A(p_0t^{-5}) &= 6\cdot 7^9t^5 + 2\cdot 7^9t^4 + 82\cdot 7^6t^3 + 156\cdot 7^4t^2 - 316\cdot 7^2t + 4\cdot 7^2\\
&\quad + p_0(-6\cdot 7^8t^5 - 15\cdot 7^7t^4 - 11\cdot 7^6t^3 - 19\cdot 7^4t^2 + 81\cdot 7^2t + 1)\\
&\quad + p_1(-6\cdot 7^9t^6 - 15\cdot 7^8t^5 - 13\cdot 7^7t^4 - 4\cdot 7^6t^3 + 41\cdot 7^3t^2 - 4\cdot 7^2t),
\end{aligned}$$

$$\begin{aligned}U_A(p_0t^{-6}) &= -6\cdot 7^9t^5 - 2\cdot 7^9t^4 - 82\cdot 7^6t^3 + 50\cdot 7^4t^2 + 3406\cdot 7^2t - 234\cdot 7\\ &\quad + p_0(7^{10}t^6 + 7^9t^5 + 15\cdot 7^7t^4 + 11\cdot 7^6t^3 + 19\cdot 7^4t^2 - 699\cdot 7^2t - 1)\\ &\quad + p_1(7^{11}t^7 + 2\cdot 7^{10}t^6 + 16\cdot 7^8t^5 + 13\cdot 7^7t^4 + 4\cdot 7^6t^3 - 453\cdot 7^3t^2\\ &\quad + 234\cdot 7t),\\ U_A(p_0t^{-7}) &= -510\cdot 7^9t^5 - 170\cdot 7^9t^4 - 6970\cdot 7^6t^3 - 17446\cdot 7^4t^2 - 35930\cdot 7^2t\\ &\quad + 258\cdot 7^2 + p_0(7^{12}t^7 + 85\cdot 7^9t^5 + 1275\cdot 7^7t^4 + 935\cdot 7^6t^3\\ &\quad + 1615\cdot 7^4t^2 + 5673\cdot 7^2t - 85) + p_1(7^{13}t^8 + 7^12t^7 + 85\cdot 7^{10}t^6\\ &\quad + 1360\cdot 7^8t^5 + 1105\cdot 7^7t^4 + 340\cdot 7^6t^3 + 4887\cdot 7^3t^2 - 258\cdot 7^2t).\end{aligned}$$

Group III

$$\begin{aligned}U_A(p_1) &= -8\cdot 7^{10}t^6 - 176\cdot 7^8t^5 - 16\cdot 7^7t^3 - 1464\cdot 7^6t^4 - 8392\cdot 7^2t^2 - 3072t\\ &\quad + p_0(-7^{12}t^7 - 23\cdot 7^{10}t^6 - 206\cdot 7^8t^5 - 125\cdot 7^7t^4 - 242\cdot 7^5t^3 - 23\cdot 7^4t^2\\ &\quad - 73t) + p_1(7^{11}t^7 + 23\cdot 7^9t^6 + 198\cdot 7^7t^5 + 109\cdot 7^6t^4 + 170\cdot 7^4t^3\\ &\quad + 439\cdot 7t^2),\\ U_A(p_1t^{-1}) &= -7p_1t,\\ U_A(p_1t^{-2}) &= 6\cdot 7^3t^2 + 90\cdot 7t - 6 - 18\cdot 7p_0t + p_1(-7^3t^3 - 12\cdot 7^2t^2 + 6t),\\ U_A(p_1t^{-3}) &= -38\cdot 7^3t^2 - 570\cdot 7t + 38 + 144\cdot 7p_0t + p_1(-7^5t^4 + 76\cdot 7^2t^2 - 38t),\\ U_A(p_1t^{-4}) &= 218\cdot 7^3t^2 + 3270\cdot 7t - 218 - 654\cdot 7p_0t\\ &\quad + p_1(-7^7t^5 - 436\cdot 7^2t^2 + 248t),\\ U_A(p_1t^{-5}) &= -6\cdot 7^9t^5 - 2\cdot 7^9t^4 - 82\cdot 7^6t^3 - 340\cdot 7^4t^2 - 2444\cdot 7^2t + 156\cdot 7\\ &\quad + p_0(7^9t^5 + 15\cdot 7^7t^4 + 11\cdot 7^6t^3 + 19\cdot 7^4t^2 + 471\cdot 7^2t - 1)\\ &\quad + p_1(6\cdot 7^9t^6 + 16\cdot 7^8t^5 + 13\cdot 7^7t^4 + 4\cdot 7^6t^3 + 327\cdot 7^3t^2 - 156\cdot 7t),\\ U_A(p_1t^{-6}) &= 21\cdot 7^9t^5 + 18\cdot 7^9t^4 + 738\cdot 7^6t^3 + 46\cdot 7^6t^2 + 9906\cdot 7^2t - 598\cdot 7\\ &\quad + p_0(-9\cdot 7^9t^5 - 135\cdot 7^7t^4 - 99\cdot 7^6t^3 - 171\cdot 7^4t^2 - 1821\cdot 7^2t + 9)\\ &\quad + p_1(-7^{11}t^7 - 9\cdot 7^{10}t^6 - 144\cdot 7^8t^5 - 117\cdot 7^7t^4 - 36\cdot 7^6t^3\\ &\quad - 1331\cdot 7^3t^2 + 598\cdot 7t).\end{aligned}$$

Group IV

$$\begin{aligned}U_B(t) &= 7^2t + 7,\\ U_B(1) &= -7,\\ U_B(t^{-1}) &= 7^2 + t^{-1},\\ U_B(t^{-2}) &= 7^5t^2 + 11\cdot 7^3t - 11\cdot 7 - 11t^{-1},\end{aligned}$$

$$U_B(t^{-3}) = 7^7t^3 - 90\cdot 7^3t - 20\cdot 7^2 + 90t^{-1},$$
$$U_B(t^{-4}) = -7^9t^4 - 38\cdot 7^7t^3 - 38\cdot 7^6t^2 - 19\cdot 7^4t + 209\cdot 7^2 - 627t^{-1},$$
$$U_B(t^{-5}) = 7^{11}t^5 + 46\cdot 7^9t^4 + 874\cdot 7^7t^3 + 874\cdot 7^6t^2 + 1955\cdot 7^4t - 667\cdot 7^2 + 3795t^{-1}.$$

Group V

$$U_B(p_0t) = 8\cdot 7^{12}t^6 + 200\cdot 7^{10}t^5 + 1984\cdot 7^8t^4 + 9656\cdot 7^6t^3 + 22896\cdot 7^4t^2 + 3144\cdot 7^3t + 632\cdot 7 + p_0(7^{14}t^7 + 26\cdot 7^{12}t^6 + 274\cdot 7^{10}t^5 + 1464\cdot 7^8t^4 + 4045\cdot 7^6t^3 + 5172\cdot 7^4t^2 + 2150\cdot 7^2t + 9\cdot 7) + p_1(-7^{13}t^7 - 26\cdot 7^{11}t^6 - 38\cdot 7^{10}t^5 - 1328\cdot 7^7t^4 - 459\cdot 7^6t^3 - 3132\cdot 7^3t^2 - 30\cdot 7^2t),$$
$$U_B(p_0) = 6\cdot 7p_0 - 7^2p_1t,$$
$$U_B(p_0t^{-1}) = -4\cdot 7^3t + 4\cdot 7 + 4t^{-1} + p_0(7^3t - 2\cdot 7) + p_1(7^4t^2 + 7^2t - 4),$$
$$U_B(p_0t^{-2}) = 8\cdot 7^5t^2 + 76\cdot 7^3t + 12\cdot 7^2 - 20t^{-1} + p_0(-7^5t^2 - 10\cdot 7^3t + 3\cdot 7) + p_1(-7^6t^3 - 9\cdot 7^4t^2 + 3\cdot 7),$$
$$U_B(p_0t^{-3}) = 8\cdot 7^7t^3 + 48\cdot 7^5t^2 - 12\cdot 7^4t - 44\cdot 7^2 + 100t^{-1} + p_0(-7^7t^3 - 8\cdot 7^5t^2 + 8\cdot 7^3t - 3\cdot 7^2) + p_1(-7^8t^4 - 7^7t^3 - 10\cdot 7^4t^2 - 2\cdot 7^4t - 17\cdot 7),$$
$$U_B(p_0t^{-4}) = -4\cdot 7^9t^4 - 164\cdot 7^7t^3 - 4\cdot 7^8t^2 - 480\cdot 7^4t - 136\cdot 7^2 - 424t^{-1} + p_0(13\cdot 7^7t^3 + 23\cdot 7^6t^2 + 55\cdot 7^4t + 31\cdot 7^2 + t^{-1}) + p_1(9\cdot 7^8t^4 + 13\cdot 7^7t^3 + 46\cdot 7^5t^2 + 135\cdot 7^3t + 94\cdot 7),$$
$$U_B(p_0t^{-5}) = 12\cdot 7^{10}t^4 + 316\cdot 7^8t^3 + 2540\cdot 7^6t^2 + 7244\cdot 7^4t + 4092\cdot 7^2 + 148\cdot 7t^{-1} + p_0(7^{11}t^5 + 13\cdot 7^9t^4 - 71\cdot 7^7t^3 - 241\cdot 7^6t^2 - 761\cdot 7^4t - 286\cdot 7^2 - 13t^{-1}) + p_1(7^{12}t^6 + 20\cdot 7^{10}t^5 + 106\cdot 7^8t^4 - 7^7t^3 - 206\cdot 7^5t^2 - 793\cdot 7^3t - 481\cdot 7).$$

Group VI

$$U_B(p_1t) = -7p_0,$$
$$U_B(p_1) = 7p_0 + p_1(7^2t + 1),$$
$$U_B(p_1t^{-1}) = 4\cdot 7^3t + 12\cdot 7 - 4t^{-1} - 4\cdot 7p_0 + p_1(-7^4t^2 - 10\cdot 7^2t - 6),$$
$$U_B(p_1t^{-2}) = -8\cdot 7^5t^2 - 100\cdot 7^3t - 36\cdot 7^2 + 44t^{-1} + p_0(2\cdot 7^5t^2 + 10\cdot 7^3t + 3\cdot 7^2) + p_1(7^6t^3 + 16\cdot 7^4t^2 + 80\cdot 7^2t + 5\cdot 7),$$
$$U_B(p_1t^{-3}) = -8\cdot 7^7t^3 + 92\cdot 7^4t + 316\cdot 7^2 - 356t^{-1} + p_0(-4\cdot 7^6t^2 - 20\cdot 7^4t$$

$$
\begin{aligned}
&- 17 \cdot 7^2) + p_1(7^8 t^4 - 2 \cdot 7^6 t^2 - 10 \cdot 7^4 t - 27 \cdot 7),\\
U_B(p_1 t^{-4}) &= 4 \cdot 7^9 t^4 + 228 \cdot 7^7 t^3 + 228 \cdot 7^6 t^2 + 152 \cdot 7^4 t - 240 \cdot 7^3 + 2432 t^{-1}\\
&+ p_0(7^9 t^4 + 15 \cdot 7^7 t^3 + 7^8 t^2 + 209 \cdot 7^4 t + 99 \cdot 7^2 - t^{-1})\\
&+ p_1(-18 \cdot 7^8 t^4 - 17 \cdot 7^7 t^3 + 26 \cdot 7^5 t^2 + 321 \cdot 7^3 t + 128 \cdot 7),\\
U_B(p_1 t^{-5}) &= -116 \cdot 7^9 t^4 - 572 \cdot 7^8 t^3 - 4604 \cdot 7^6 t^2 - 11804 \cdot 7^4 t + 1444 \cdot 7^2\\
&- 2036 \cdot 7 t^{-1} + p_0(-2 \cdot 7^{11} t^5 - 57 \cdot 7^9 t^4 - 79 \cdot 7^8 t^3 - 89 \cdot 7^7 t^2\\
&- 1977 \cdot 7^4 t - 584 \cdot 7^2 + 3 \cdot 7 t^{-1}) + p_1(-7^{12} t^6 - 19 \cdot 7^{10} t^5 + 74 \cdot 7^8 t^4\\
&+ 209 \cdot 7^7 t^3 + 470 \cdot 7^5 t^2 - 709 \cdot 7^3 t - 465 \cdot 7).
\end{aligned}
$$

Appendix B. The Fundamental Relations for the crank parity function for powers of 7

Group I

$$
\begin{aligned}
U^{(1)}(1) &= 2p_0 + 7p_1,\\
U^{(1)}(t^{-1}) &= 7^2 t + p_0(-4 \cdot 7^2 t - 4) + 4 \cdot 7 p_1,\\
U^{(1)}(t^{-2}) &= 7^4 t^2 - 4 \cdot 7^2 t + 1 + p_0(-4 \cdot 7^4 t^2 + 8 \cdot 7^2 t + 2 \cdot 7) + p_1(4 \cdot 7^3 t^2 - 13 \cdot 7),\\
U^{(1)}(t^{-3}) &= -8 \cdot 7^4 t^2 + 24 \cdot 7^2 t - 2 \cdot 7 + p_0(2 \cdot 7^6 t^6 + 8 \cdot 7^5 t^2 + 8 \cdot 7^3 t - 4 \cdot 7)\\
&+ p_1(-2 \cdot 7^5 t^2 - 55 \cdot 7^3 t + t^{-1}),\\
U^{(1)}(t^{-4}) &= -7^8 t^4 - 2 \cdot 7^7 t^3 - 2 \cdot 7^5 t^2 - 32 \cdot 7^3 t + 20 \cdot 7\\
&+ p_0(-4 \cdot 7^7 t^3 - 76 \cdot 7^5 t^2 - 144 \cdot 7^3 t - 4 \cdot 7^2)\\
&+ p_1(4 \cdot 7^6 t^2 + 74 \cdot 7^4 t + 78 \cdot 7^2 - 2 \cdot 7 t^{-1}),\\
U^{(1)}(t^{-5}) &= -7^{10} t^5 + 4 \cdot 7^8 t^4 + 27 \cdot 7^7 t^3 + 116 \cdot 7^5 t^2 + 314 \cdot 7^3 t - 27 \cdot 7^2\\
&+ p_0(4 \cdot 7^{10} t^5 - 64 \cdot 7^8 t^4 + 90 \cdot 7^7 t^3 + 712 \cdot 7^5 t^2 + 1504 \cdot 7^3 t + 492 \cdot 7)\\
&+ p_1(-4 \cdot 7^9 t^4 - 58 \cdot 7^7 t^3 - 78 \cdot 7^6 t^2 - 639 \cdot 7^4 t - 974 \cdot 7^2 + 135 t^{-1}),\\
U^{(1)}(t^{-6}) &= 7^{12} t^6 + 44 \cdot 7^{10} t^5 + 284 \cdot 7^8 t^4 - 118 \cdot 7^7 t^3 - 1348 \cdot 7^5 t^2 - 2740 \cdot 7^3 t\\
&+ 1243 \cdot 7 + p_0(-88 \cdot 7^{10} t^5 - 200 \cdot 7^9 t^4 - 204 \cdot 7^8 t^3 - 6568 \cdot 7^5 t^2\\
&- 13064 \cdot 7^3 t - 682 \cdot 7^2) + p_1(86 \cdot 7^9 t^4 + 176 \cdot 7^8 t^3 + 1130 \cdot 7^6 t^2\\
&+ 734 \cdot 7^5 t + 8679 \cdot 7^2 - 22 \cdot 7^2 t^{-1}).
\end{aligned}
$$

Group II

$$
\begin{aligned}
U^{(1)}(p_0) &= 7^{14} t^7 + 22 \cdot 7^{12} t^6 + 190 \cdot 7^{10} t^5 + 16 \cdot 7^{10} t^4 + 1497 \cdot 7^6 t^3 + 1028 \cdot 7^4 t^2\\
&+ 2 \cdot 7^4 t + p_0(8 \cdot 7^{12} t^6 + 24 \cdot 7^{11} t^5 + 192 \cdot 7^9 t^4 + 4888 \cdot 7^6 t^3
\end{aligned}
$$

$$
\begin{aligned}
&+ 7408\cdot 7^4t^2 + 2967\cdot 7^2t + 20) + p_1(7^{15}t^7 + 30\cdot 7^{13}t^6 + 366\cdot 7^{11}t^5\\
&+ 328\cdot 7^{10}t^4 + 1095\cdot 7^8t^3 + 12556\cdot 7^5t^2 + 7722\cdot 7^3t + 680\cdot 7),\\
U^{(1)}(p_0t^{-1}) &= 5\cdot 72t + p_0(-7^4t^2 - 32\cdot 7^2t + 62) + p_1(7^3t + 80\cdot 7),\\
U^{(1)}(p_0t^{-2}) &= 7^5t^2 + 25\cdot 7^2t + p_0(7^6t^3 - 16\cdot 7^4t^2 - 96\cdot 7^2t - 48)\\
&\quad + p_1(-7^5t^2 + 17\cdot 7^3t + 72\cdot 7),\\
U^{(1)}(p_0t^{-3}) &= -7^6t^3 - 14\cdot 7^4t^2 - 4\cdot 7^3t + 7 + p_0(-7^8t^4 - 4\cdot 7^6t^3 + 96\cdot 7^4t^2\\
&\quad + 472\cdot 7^2t + 4\cdot 7^2) + p_1(7^7t^3 + 3\cdot 7^5t^2 - 2\cdot 7^5t - 51\cdot 7^2 + 6t^{-1}),\\
U^{(1)}(p_0t^{-4}) &= -5\cdot 7^8t^4 - 9\cdot 7^7t^3 - 40\cdot 7^5t^2 - 66\cdot 7^3t - 9\cdot 7\\
&\quad + p_0({}_7^{10}t^5 + 24\cdot 7^8t^4 + 16\cdot 7^7t^3 - 48\cdot 7^5t^2 - 362\cdot 7^3t - 164\cdot 7)\\
&\quad + p_1(-7^9t^4 - 23\cdot 7^7t^3 - 12\cdot 7^6t^2 + 62\cdot 7^4t + 291\cdot 7^2 - 40t^{-1}),\\
U^{(1)}(p_0t^{-5}) &= -5\cdot 7^{10}t^5 - 15\cdot 7^8t^4 + 60\cdot 7^7t^3 + 458\cdot 7^5t^2 + 786\cdot 7^3t + 78\cdot 7\\
&\quad + p_0({}_7^{12}t^6 + 32\cdot 7^{10}t^5 + 44\cdot 7^9t^4 + 240\cdot 7^7t^3 + 916\cdot 7^5t^2\\
&\quad + 2416\cdot 7^3t + 1004\cdot 7) + p_1(-7^{11}t^5 - 31\cdot 7^9t^4 - 272\cdot 7^7t^3\\
&\quad - 206\cdot 7^6t^2 - 110\cdot 7^5t - 1810\cdot 7^2 + 216t^{-1}),\\
U^{(1)}(p_0t^{-6}) &= 6\cdot 7^{12}t^6 + 204\cdot 7^{10}t^5 + 240\cdot 7^9t^4 + 401\cdot 7^7t^3 - 2528\cdot 7^5t^2\\
&\quad - 5821\cdot 7^3t - 96\cdot 7^2 + p_0(-16\cdot 7^{12}t^6 - 584\cdot 7^{10}t^5 - 6576\cdot 7^8t^4\\
&\quad - 4885\cdot 7^7t^3 - 11832\cdot 7^5t^2 - 16512\cdot 7^3t - 5736\cdot 7 + t^{-1})\\
&\quad + p_1(16\cdot 7^{11}t^5 + 562\cdot 7^9t^4 + 837\cdot 7^8t^3 + 82\cdot 7^8t^2 + 1213\cdot 7^5t\\
&\quad + 1556\cdot 7^3 - 136\cdot 7t^{-1}),
\end{aligned}
$$

Group III

$$
\begin{aligned}
U^{(1)}(p_1t^{-1}) &= 7^{15}t^7 + 22\cdot 7^{13}t^6 + 190\cdot 7^{11}t^5 + 16\cdot 7^{11}t^4 + 1497\cdot 7^7t^3 + 1028\cdot 7^5t^2\\
&\quad + 687\cdot 7^2t + p_0(8\cdot 7^{13}t^6 + 24\cdot 7^{12}t^5 + 192\cdot 7^{10}t^4 + 4888\cdot 7^7t^3\\
&\quad + 7408\cdot 7^5t^2 + 20764\cdot 7^2t + 148) + p_1(30\cdot 7^{14}t^6 + 366\cdot 7^{12}t^5\\
&\quad + 328\cdot 7^{11}t^4 + 1095\cdot 7^9t^3 + 12556\cdot 7^6t^2 + 7722\cdot 7^4t + 4772\cdot 7),\\
U^{(1)}(p_1t^{-2}) &= 7^4t^2 + 36\cdot 7^2t + p_0(-12\cdot 7^4t^2 - 240\cdot 7^2t + 430)\\
&\quad + p_1(12\cdot 7^3t + 570\cdot 7),\\
U^{(1)}(p_1t^{-3}) &= 48\cdot 7^4t^2 + 193\cdot 7^2t + p_0(10\cdot 7^6t^3 - 64\cdot 7^4t^2 - 552\cdot 7^2t - 44\cdot 7)\\
&\quad + p_1(-10\cdot 7^5t^2 + 74\cdot 7^3t + 430\cdot 7 + t^{-1}),\\
U^{(1)}(p_1t^{-4}) &= -7^8t^4 - 22\cdot 7^6t^3 - 197\cdot 7^4t^2 - 64\cdot 7^3t + 8\cdot 7\\
&\quad + p_0(-8\cdot 7^8t^4 - 68\cdot 7^6t^3 + 268\cdot 7^4t^2 + 48\cdot 7^4t + 156\cdot 7)
\end{aligned}
$$

$$
\begin{aligned}
&+ p_1(8\cdot 7^7t^3 + 61\cdot 7^5t^2 - 314\cdot 7^3t - 270\cdot 7^2 + 34t^{-1}),\\
U^{(1)}(p_1t^{-5}) = &-7^{10}t^5 - 36\cdot 7^8t^4 - 44\cdot 7^7t^3 - 144\cdot 7^5t^2 - 138\cdot 7^3t - 76\cdot 7\\
&+ p_0(12\cdot 7^{10}t^5 + 264\cdot 7^8t^4 + 234\cdot 7^7t^3 + 248\cdot 7^5t^2 - 1440\cdot 7^3t\\
&- 804\cdot 7) + p_1(-12\cdot 7^9t^4 - 251\cdot 7^7t^3 - 194\cdot 7^6t^2 - 59\cdot 7^4t\\
&+ 1342\cdot 7^2 - 225t^{-1}),\\
U^{(1)}(p_1t^{-6}) = &\, 7^{12}t^6 + 4\cdot 7^{10}t^5 + 197\cdot 7^8t^4 + 414\cdot 7^7t^3 + 2196\cdot 7^5t^2 + 3124\cdot 7^3t\\
&+ 657\cdot 7 + p_0(8\cdot 7^{12}t^6 + 160\cdot 7^{10}t^5 + 808\cdot 7^8t^4 + 204\cdot 7^7t^3\\
&+ 1144\cdot 7^5t^2 + 8776\cdot 7^3t + 638\cdot 7^2) + p_1(-8\cdot 7^11t^5 - 153\cdot 7^9t^4\\
&- 652\cdot 7^7t^3 - 158\cdot 7^6t^2 - 178\cdot 7^5t - 7619\cdot 7^2 + 1170t^{-1}),\\
U^{(1)}(p_1t^{-7}) = &-7^{14}t^7 - 2\cdot 7^{12}t^6 + 563\cdot 7^{10}t^5 + 740\cdot 7^9t^4 + 676\cdot 7^7t^3\\
&- 12972\cdot 7^5t^2 - 26004\cdot 7^3t - 776\cdot 7^2 + p_0(2\cdot 7^{14}t^7 - 88\cdot 7^{12}t^6\\
&- 3180\cdot 7^{10}t^5 - 4496\cdot 7^9t^4 - 20086\cdot 7^7t^3 - 40840\cdot 7^5t^2\\
&- 60792\cdot 7^3t - 3340\cdot 7^2 + 10t^{-1}) + p_1(-2\cdot 7^{13}t^6 + 13\cdot 7^{12}t^5\\
&+ 3084\cdot 7^9t^4 + 3986\cdot 7^8t^3 + 16260\cdot 7^6t^2 + 28524\cdot 7^4t + 43476\cdot 7^2\\
&- 2\cdot 7^4t^{-1}).
\end{aligned}
$$

Group IV

$$
\begin{aligned}
&U^{(0)}(1) = 1,\\
&U^{(0)}(t^{-1}) = -7t - 4,\\
&U^{(0)}(t^{-2}) = -7^3t^2 + 20,\\
&U^{(0)}(t^{-3}) = -7^5t^3 - 88,\\
&U^{(0)}(t^{-4}) = -7^7t^4 + 260,\\
&U^{(0)}(t^{-5}) = -7^9t^5 + 68\cdot 7,\\
&U^{(0)}(t^{-6}) = -7^11t^6 - 2392\cdot 7.
\end{aligned}
$$

Group V

$$
\begin{aligned}
U^{(0)}(p_0) = &\, 7^9t^5 + 11\cdot 7^7t^4 + 38\cdot 7^5t^3 + 31\cdot 7^3t^2 + 6\cdot 7t\\
&+ p_0(8\cdot 7^7t^4 + 80\cdot 7^5t^3 + 216\cdot 7^3t^2 + 79\cdot 7t)\\
&+ p_1(7^{10}t^5 + 19\cdot 7^8t^4 + 18\cdot 7^7t^3 + 327\cdot 7^4t^2 + 34\cdot 7^3t + 4),\\
U^{(0)}(p_0t^{-1}) = &-7^2t + 1 + p_0(-7^3t^2 - 8\cdot 7t) + p_1(7^2t + 7),\\
U^{(0)}(p_0t^{-2}) = &-7^4t^2 - 7^3t - 12 + p_0(-7^5t^3 - 8\cdot 7^3t^2) + p_1(7^4t^2 + 7^3t),
\end{aligned}
$$

$$\begin{aligned}
U^{(0)}(p_0t^{-3}) &= 2\cdot 7^6t^3 + 24\cdot 7^4t^2 + 83\cdot 7^2t + 108 + p_0(6\cdot 7^7t^4 + 62\cdot 7^5t^3\\
&\quad + 27\cdot 7^4t^2 + 10\cdot 7^2t + 1) + p_1(-6\cdot 7^6t^3 - 8\cdot 7^5t^2 - 3\cdot 7^4t - 4\cdot 7),\\
U^{(0)}(p_0t^{-4}) &= -7^8t^4 - 25\cdot 7^6t^3 - 186\cdot 7^4t^2 - 498\cdot 7^2t - 120\cdot 7\\
&\quad + p_0(-7^9t^5 - 50\cdot 7^7t^4 - 60\cdot 7^6t^3 - 162\cdot 7^4t^2 - 60\cdot 7^2t - 6)\\
&\quad + p_1(7^8\cdot t^4 + 7^8t^3 + 54\cdot 7^5t^2 + 18\cdot 7^4t + 24\cdot 7),\\
U^{(0)}(p_0t^{-5}) &= -7^{10}t^5 - 7^9t^4 + 9\cdot 7^7t^3 + 93\cdot 7^5t^2 + 249\cdot 7^3t + 836\cdot 7\\
&\quad + p_0(-7^{11}t^6 - 8\cdot 7^9t^5 + 3\cdot 7^9t^4 + 30\cdot 7^7t^3 + 81\cdot 7^5t^2 + 30\cdot 7^3t\\
&\quad + 3\cdot 7) + p_1(7^{10}t^5 + 7^9t^4 - 3\cdot 7^8t^3 - 27\cdot 7^6t^2 - 9\cdot 7^5t - 12\cdot 7^2),\\
U^{(0)}(p_0t^{-6}) &= -7^{12}t^6 - 7^{11}t^5 + 6\cdot 7^7t^3 + 62\cdot 7^5t^2 + 166\cdot 7^3t - 748\cdot 7^2\\
&\quad + p_0(-7^{13}t^7 - 8\cdot 7^{11}t^6 + 2\cdot 7^9t^4 + 20\cdot 7^7t^3 + 54\cdot 7^5t^2 + 20\cdot 7^3t\\
&\quad + 2\cdot 7) + p_1(7^{12}t^6 + 7^{11}t^5 - 2\cdot 7^8t^3 - 18\cdot 7^6t^2 - 6\cdot 7^5t - 8\cdot 7^2),
\end{aligned}$$

Group VI

$$\begin{aligned}
U^{(0)}(p_1t^{-1}) &= 7^{10}t^5 + 11\cdot 7^8t^4 + 38\cdot 7^6t^3 + 31\cdot 7^4t^2 + 41\cdot 7t - 1,\\
&\quad + p_0(8\cdot 7^8t^4 + 80\cdot 7^6t^3 + 216\cdot 7^4t^2 + 552\cdot 7t)\\
&\quad + p_1(7^{11}t^5 + 19\cdot 7^9t^4 + \cdot 7^8t^3 + \cdot 7^5t^2 + \cdot 7^4t + 29),\\
U^{(0)}(p_1t^{-2}) &= -7^3t^2 - 8\cdot 7^2t + 11 + p_0(-8\cdot 7^3t^2 - 8\cdot 7^2t) + p_1(8\cdot 7^2t + 7^2),\\
U^{(0)}(p_1t^{-3}) &= -7^5t^3 - 8\cdot 7^4t^2 - 7^4t - 96 + p_0(-8\cdot 7^5t^3 - 8\cdot 7^4t^2)\\
&\quad + p_1(8\cdot 7^4t^2 + 7^4t),\\
U^{(0)}(p_1t^{-4}) &= -7^7t^4 + 16\cdot 7^6t^3 + 199\cdot 7^4t^2 + 664\cdot 7^2t + 740\\
&\quad + p_0(48\cdot 7^7t^4 + 72\cdot 7^6t^3 + 216\cdot 7^4t^2 + 80\cdot 7^2t + 8)\\
&\quad + p_1(-48\cdot 7^6t^3 - 65\cdot 7^5t^2 - 24\cdot 7^4t - 32\cdot 7),\\
U^{(0)}(p_1t^{-5}) &= -7^9t^5 - 8\cdot 7^8t^4 - 31\cdot 7^7t^3 - 248\cdot 7^5t^2 - 664\cdot 7^3t - 740\cdot 7\\
&\quad + p_0(-8\cdot 7^9t^5 - 64\cdot 7^8t^4 - 80\cdot 7^7t^3 - 216\cdot 7^5t^2 - 80\cdot 7^3t - 8\cdot 7)\\
&\quad + p_1(8\cdot 7^8t^4 + 9\cdot 7^8t^3 + 72\cdot 7^6t^2 + 24\cdot 7^5t + 32\cdot 7^2),\\
U^{(0)}(p_1t^{-6}) &= -7^{11}t^6 - 8\cdot 7^{10}t^5 - 7^{10}t^4 + 120\cdot 7^7t^3 + 1240\cdot 7^5t^2 + 3320\cdot 7^3t\\
&\quad + 4720\cdot 7 + p_0(-8\cdot 7^{11}t^6 - 8\cdot 7^{10}t^5 + 40\cdot 7^9t^4 + 400\cdot 7^7t^3\\
&\quad + 1080\cdot 7^5t^2 + 400\cdot 7^3t + 40\cdot 7) + p_1(8\cdot 7^{10}t^5 + 7^{10}t^4 - 40\cdot 7^8t^3\\
&\quad - 360\cdot 7^6t^2 - 120\cdot 7^5t - 160\cdot 7^2),\\
U^{(0)}(p_1t^{-7}) &= 6\cdot 7^{13}t^7 + 20\cdot 7^{12}t^6 + 272\cdot 7^{10}t^5 + 1893\cdot 7^8t^4 + 435\cdot 7^7t^3
\end{aligned}$$

$$- 2953 \cdot 7^5 t^2 - 10620 \cdot 7^3 t - 27029 \cdot 7 + p_0(-8 \cdot 7^{13} t^7 - 8 \cdot 7^{12} t^6$$
$$- 136 \cdot 7^9 t^4 - 1360 \cdot 7^7 t^3 - 3672 \cdot 7^5 t^2 - 1360 \cdot 7^3 t - 136 \cdot 7)$$
$$+ p_1(8 \cdot 7^{12} t^6 + 6 \cdot 7^{11} t^5 - 19 \cdot 7^9 t^4 + 818 \cdot 7^7 t^3 + 1167 \cdot 7^6 t^2$$
$$+ 2798 \cdot 7^4 t + 11 \cdot 7^4 + t^{-1}).$$

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