NEW FIFTH AND SEVENTH ORDER MOCK THETA FUNCTION IDENTITIES

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Dedicated to George Andrews on the occasion of his eightieth birthday

Abstract. We give simple proofs of Hecke-Rogers indefinite binary theta series identities for the two Ramanujan fifth order mock theta functions $\chi_0(q)$ and $\chi_1(q)$ and all three of Ramanujan's seventh order mock theta functions. We find that the coefficients of the three mock theta functions of order 7 are surprisingly related.

1. Introduction

In his last letter to G. H. Hardy, Ramanujan described new functions that he called mock theta functions and listed mock theta functions of order 3, 5 and 7. Watson studied the behaviour of the third order functions under the modular group, but was unable to find similar transformation properties for the fifth and seventh order functions. The first substantial progress towards finding such transformation properties was made by Andrews [1], who found double sum representations for the fifth and seventh order functions. These double sum representations were reminiscent of certain identities for modular forms found by Hecke and Rogers. Andrews results for the fifth and seventh order mock theta functions were crucial to Zwegers [14], who later showed how to complete these functions to harmonic Maass forms. For more details on this aspect see Zagier's survey [12].

Throughout this paper we use following standard notation:

$$(a;q)_\infty = (a)_\infty = \prod_{n=1}^{\infty}(1-aq^n),$$

$$(a;q)_n = (a)_n = (a;q)_\infty/(aq^n;q)_\infty$$
$$\big(= (1-a)(1-aq)\cdots(1-aq^{n-1}) \quad \text{for } n \text{ a nonnegative integer}\big).$$

Andrews [1] found Hecke-Rogers indefinite binary theta series identities for all the fifth order mock theta functions except for the following two:

$$\chi_0(q) = \sum_{n=0}^{\infty}\frac{q^n}{(q^{n+1};q)_n} = \sum_{n=0}^{\infty}\frac{q^n\,(q)_n}{(q)_{2n}}$$
$$= 1 + q + q^2 + 2\,q^3 + q^4 + 3\,q^5 + 2\,q^6 + 3\,q^7 + \cdots,$$

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and

$$\chi_1(q) = \sum_{n=0}^{\infty} \frac{q^n}{(q^{n+1};q)_{n+1}} = \sum_{n=0}^{\infty} \frac{q^n\,(q)_n}{(q)_{2n+1}}$$
$$= 1 + 2\,q + 2\,q^2 + 3\,q^3 + 3\,q^4 + 4\,q^5 + 4\,q^6 + 6\,q^7 + \cdots .$$

Zwegers [13] found triple sum identities for $\chi_0(q)$ and $\chi_1(q)$. Zagier [12] stated indefinite binary theta series identities for these two functions but gave few details. We find new Hecke-Rogers indefinite binary theta series identities for these two functions. In Section 5 we compare our results with Zagier's.

Theorem 1.1.

$$\begin{aligned} &(q)_\infty(\chi_0(q)-2) \\ (1.1)\quad &= \sum_{j=0}^{\infty} \sum_{-j\le 3m\le j} \operatorname{sgn}(m)(-1)^{m+j+1} q^{j(3j+1)/2-m(15m+1)/2}(1-q^{2j+1}) \\ &+ \sum_{j=1}^{\infty} \sum_{-j-1\le 3m\le j-1} \operatorname{sgn}(m)(-1)^{m+j+1} q^{j(3j+1)/2-m(15m+11)/2-1}(1-q^{2j+1}), \end{aligned}$$

and

$$\begin{aligned} &(q)_\infty\chi_1(q) \\ (1.2)\quad &= \sum_{j=1}^{\infty} \sum_{-j\le 3m\le j-1} \operatorname{sgn}(m)(-1)^{m+j+1} q^{j(3j-1)/2-m(15m+7)/2-1}(1+q^{j}) \\ &+ \sum_{j=1}^{\infty} \sum_{-j-1\le 3m\le j-2} \operatorname{sgn}(m)(-1)^{m+j+1} q^{j(3j-1)/2-m(15m+13)/2-2}(1+q^{j}), \end{aligned}$$

where

$$\operatorname{sgn}(m) = \begin{cases} 1 & \text{if } m \ge 0, \\ -1 & \text{if } m < 0. \end{cases}$$

Idea of Proof. We need the following conjugate Bailey pair (with $a = q$):

$$\delta_n = \frac{q^n(q)_n(q)_\infty}{(1-q)},$$
$$\gamma_n = \sum_{j=n+1}^{\infty} (-1)^{j+n+1} q^{j(3j-1)/2-3n(n+1)/2-1}(1+q^j).$$

The proof of this only uses Heine's transformation [5, Eq.(III.I)] and an exercise from Andrews's book [2, Ex.10,p.29]. The rest of the proof of Theorem 1.1 uses this conjugate Bailey pair, the Bailey transform and Slater's Bailey pairs A(4) and A(2) (with $a = q$) [8, p.463]. The necessary background on conjugate Bailey pairs, Bailey pairs and the Bailey transform is given in Section 2. In Section 3 the proof of Theorem 1.1 is completed.

Using the same conjugate Bailey pair and Slater's A(7*), A(8) and A(6) (with $a = q$) lead to new Hecke-Rogers indefinite binary theta series identities for Ramanujan's three seventh order mock theta functions. A(7*) is actually a variant of A(7) adjusted to work with $a = q$ instead of $a = 1$. The three identities given below in Theorem 1.2 appear to be new. The following are Ramanujan's three seventh order mock theta functions:

$$\begin{aligned}
\mathcal{F}_0(q) &= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^{n+1};q)_n} \\
&= 1 + q + q^3 + q^4 + q^5 + 2\,q^7 + q^8 + 2\,q^9 + \cdots , \\
\mathcal{F}_1(q) &= \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q^{n};q)_n} \\
&= q + q^2 + q^3 + 2\,q^4 + q^5 + 2\,q^6 + 2\,q^7 + 2\,q^8 + \cdots , \\
\mathcal{F}_2(q) &= \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q^{n+1};q)_{n+1}} \\
&= 1 + q + 2\,q^2 + q^3 + 2\,q^4 + 2\,q^5 + 3\,q^6 + 2\,q^7 + \cdots .
\end{aligned}$$

We have the following theorem

Theorem 1.2.

$$\begin{aligned}
&(q)_\infty \mathcal{F}_0(q) \\
&= \sum_{j=1}^{\infty} \sum_{-j \le 3m \le j-1} \operatorname{sgn}(m)(-1)^{m+j+1} q^{j(3j-1)/2 - m(21m+13)/2-1} \\
&\qquad (1+q^j)(1-q^{6m+1}), \qquad (1.3)
\end{aligned}$$

$$\begin{aligned}
&(q)_\infty \mathcal{F}_1(q) \\
&= \sum_{j=1}^{\infty} \sum_{-j \le 3m \le j-1} \operatorname{sgn}(m)(-1)^{m+j+1} q^{j(3j-1)/2 - m(21m+5)/2}(1+q^j) \qquad (1.4) \\
&+ \sum_{j=2}^{\infty} \sum_{-j-1 \le 3m \le j-2} \operatorname{sgn}(m)(-1)^{m+j+1} q^{j(3j-1)/2 - m(21m+19)/2-2}(1+q^j)
\end{aligned}$$

$$\begin{aligned}
&(q)_\infty \mathcal{F}_2(q) \\
&= \sum_{j=1}^{\infty} \sum_{-j \le 3m \le j-1} \operatorname{sgn}(m)(-1)^{m+j+1} q^{j(3j-1)/2 - m(21m+11)/2-1}(1+q^j) \qquad (1.5) \\
&+ \sum_{j=2}^{\infty} \sum_{-j-1 \le 3m \le j-2} \operatorname{sgn}(m)(-1)^{m+j+1} q^{j(3j-1)/2 - m(21m+17)/2-2}(1+q^j).
\end{aligned}$$

We prove this theorem in Section 4. In his last letter to Hardy, all that Ramanujan said about the seventh order functions was that there were not related to each other. Surprisingly we show that the coefficients of the three seventh order functions are indeed related, although this is probably not the kind of relationship that Ramanujan had in mind. For example we find for $n \ge 0$ that

$$f_0(25n+8) = f_2(n), \qquad (1.6)$$

$$f_1(25n+1) = f_0(n), \qquad (1.7)$$

$$f_2(25n-3) = -f_1(n), \qquad (1.8)$$

where we define $f_j(n)$ by

$$\sum_{n=0}^{\infty} f_j(n)q^n = (q)_\infty \, \mathcal{F}_j(q),$$

for $j = 0,\, 1,\, 2$. This and more general results including analogous results for the fifth order functions are proved in Section 5.

2. The Bailey Transform and Conjugate Bailey Pairs

Theorem 2.1 (The Bailey Transform)**.** *Subject to suitable convergence conditions, if*

$$\beta_n = \sum_{r=0}^{n} \alpha_r u_{n-r} v_{n+r}, \quad \text{and} \qquad \gamma_n = \sum_{r=n}^{\infty} \delta_r u_{r-n} v_{r+n}, \tag{2.1}$$

then

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n. \tag{2.2}$$

When applying his transform, Bailey [4] chose $u_n = 1/(q)_n$ and $v_n = 1/(aq;q)_n$. This motivates the following definitions:

Definition 2.2. A pair of sequences (α_n, β_n) is a **Bailey pair** relative to (a, q) if

$$\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(q)_{n-r}(aq)_{n+r}}, \tag{2.3}$$

for $n \geq 0$.

Definition 2.3. A pair of sequences (γ_n, δ_n) is a **conjugate Bailey pair** relative to (a, q) if

$$\gamma_n = \sum_{r=n}^{\infty} \frac{\delta_r}{(q)_{r-n}(aq)_{r+n}}, \tag{2.4}$$

for $n \geq 0$.

The basic idea is to find a suitable conjugate Bailey pair and apply the Bailey Transform using known Bailey pairs.

Theorem 2.4. *The sequences*

$$\delta_n = \frac{q^n (q)_n (q)_\infty}{(1-q)}, \tag{2.5}$$

$$\gamma_n = \sum_{j=n+1}^{\infty} (-1)^{j+n+1} q^{j(3j-1)/2 - 3n(n+1)/2 - 1}(1 + q^j), \tag{2.6}$$

form a conjugate Bailey pair relative to (q, q); i.e. $a = q$.

Remark 2.5. We note that this result can be deduced from a special case of a result of Lovejoy [7, Thm1.1(4),p.53]. We give a simple proof that uses only Heine's transformation and a combinatorial result of Andrews [2, Ex.10,p.29].

Proof. We let

$$\delta_n = (q)_n (q)_\infty \frac{q^n}{(1-q)},$$

and

$$\gamma_n = \sum_{r=n}^{\infty} \frac{\delta_r}{(q)_{r-n}(q^2;q)_{r+n}} = (q)_\infty \sum_{r=n}^{\infty} \frac{(q)_r q^r}{(q)_{r-n}(q;q)_{r+n+1}}.$$

We must show that γ_n is given by (2.6).

$$\begin{aligned}
\sum_{r=n}^{\infty} \frac{(q)_r q^r}{(q)_{r-n}(q;q)_{r+n+1}} &= \sum_{r=0}^{\infty} \frac{(q)_{r+n} q^{r+n}}{(q)_r (q)_{r+2n+1}} \\
&= q^n \frac{(q)_n}{(q)_{2n+1}} \sum_{r=0}^{\infty} \frac{(q^{n+1};q)_r q^r}{(q)_r (q^{2n+2};q)_r} = q^n \frac{(q)_n}{(q)_{2n+1}} {}_2\phi_1 \left(\begin{matrix} 0, & q^{n+1}; & q, & q \\ & q^{2n+2} & & \end{matrix} \right) \\
&= q^n \frac{(q)_n}{(q)_{2n+1}} \frac{(q^{n+1};q)_\infty}{(q^{2n+2};q)_\infty (q)_\infty} {}_2\phi_1 \left(\begin{matrix} q^{n+1}, & q; & q, & q^{n+1} \\ & 0 & & \end{matrix} \right) \\
&= q^n \frac{1}{(q)_\infty} \sum_{j=0}^{\infty} (q^{n+1};q)_j q^{(n+1)j},
\end{aligned}$$

by Heine's transformation [5, Eq.(III.I)], so that

$$\gamma_n = q^n \sum_{j=0}^{\infty} (q^{n+1};q)_j q^{(n+1)j}. \tag{2.7}$$

From Andrews [2, Ex.10,p.29] we have

$$\sum_{j=0}^{\infty} (xq)_j x^{j+1} q^{j+1} = \sum_{m=1}^{\infty} (-1)^{m-1} q^{m(3m-1)/2} x^{3m-2} (1 + xq^m). \tag{2.8}$$

Using (2.7) and (2.8) with $x = q^n$ we have

$$\begin{aligned}
\gamma_n &= q^n \sum_{j=0}^{\infty} (q^{n+1};q)_j q^{(n+1)j} \\
&= \sum_{m=1}^{\infty} (-1)^{m-1} q^{m(3m-1)/2 + n(3m-2) - 1} (1 + q^{m+n}) \\
&= \sum_{m=n+1}^{\infty} (-1)^{m+n+1} q^{m(3m-1)/2 - 3n(n+1)/2 - 1} (1 + q^m),
\end{aligned}$$

as required. We note that Subbarao [9] gave a combinatorial proof of (2.8) by using a variant of Franklin's involution [2, pp.10–11]. □

3. Proof of Theorem 1.1

To prove Theorem 1.1 we will apply the Bailey Transform, with $u_n = 1/(q)_n$, $v_n = 1/(q^2;q)_n$, using the conjugate Bailey pair in Theorem 2.4, and Slater's Bailey pairs $A(4)$ and $A(2)$. By [8, p.463], the following gives Slater's $A(4)$ Bailey pair relative to (q,q):

$$\beta_n = \frac{q^n}{(q^2;q)_{2n}}, \qquad \alpha_n = \begin{cases} q^{6m^2-4m} & \text{if } n=3m-1, \\ q^{6m^2+4m} & \text{if } n=3m, \\ -q^{6m^2+8m+2} - q^{6m^2+4m} & \text{if } n=3m+1. \end{cases} \tag{3.1}$$

By [11, Eq.(A_0),p.278] we have

$$\begin{aligned} \chi_0(q) &= \sum_{n=0}^{\infty} \frac{q^n}{(q^{n+1};q)_n} = 1 + \sum_{n=0}^{\infty} \frac{q^{2n+1}}{(q^{n+1};q)_{n+1}} \\ &= 1 + q\sum_{n=0}^{\infty} \frac{q^n}{(q^2;q)_{2n}} \cdot \frac{q^n(q)_n}{(1-q)} = 1 + \frac{q}{(q)_\infty}\sum_{n=0}^{\infty} \beta_n\delta_n, \end{aligned}$$

where δ_n is given in (2.5). Thus by the Bailey Transform and (3.1) we have

(3.2)

$$\begin{aligned} &(q)_\infty\left(\chi_0(q)-1\right) = q\sum_{n=0}^{\infty}\beta_n\delta_n = q\sum_{n=0}^{\infty}\alpha_n\gamma_n \\ &= \sum_{m=1}^{\infty}\sum_{j=3m}^{\infty}(-1)^{m+j}q^{j(3j-1)/2-m(15m-1)/2}(1+q^j) \\ &\quad + \sum_{m=0}^{\infty}\sum_{j=3m+1}^{\infty}(-1)^{m+j+1}q^{j(3j-1)/2-m(15m+1)/2}(1+q^j) \\ &\quad + \sum_{m=0}^{\infty}\sum_{j=3m+2}^{\infty}(-1)^{m+j+1}\Big\{q^{j(3j-1)/2-m(15m+11)/2-1} \\ &\qquad\qquad + q^{j(3j-1)/2-m(15m+19)/2-3}\Big\}(1+q^j) \\ &= \sum_{j=1}^{\infty}\sum_{-j\le 3m\le j-1}\operatorname{sgn}(m)(-1)^{m+j+1}q^{j(3j-1)/2-m(15m+1)/2}(1+q^j) \\ &\quad + \sum_{j=2}^{\infty}\sum_{-j-1\le 3m\le j-2}\operatorname{sgn}(m)(-1)^{m+j+1}q^{j(3j-1)/2-m(15m+11)/2-1}(1+q^j), \end{aligned}$$

by noting that

$$-(-m-1)(15(-m-1)+19)/2-3 = -m(15m+11)/2-1.$$

Now from Euler's Pentagonal Number Theorem [2, p.11] we have

$$(q)_\infty = \sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n-1)/2} = \sum_{m=-\infty}^{\infty} q^{6m^2+m} - \sum_{m=-\infty}^{\infty} q^{6m^2+5m+1}. \tag{3.3}$$

By (3.2) and (3.3) we have

$$\begin{aligned} &(q)_\infty\left(\chi_0(q)-2\right) = (q)_\infty\left(\chi_0(q)-1\right) - (q)_\infty \\ &= \sum_{j=1}^{\infty}\sum_{-j\le 3m\le j-1}\operatorname{sgn}(m)(-1)^{m+j+1}q^{j(3j-1)/2-m(15m+1)/2}(1+q^j) \\ &\quad + \sum_{j=1}^{\infty}\sum_{-j-1\le 3m\le j-2}\operatorname{sgn}(m)(-1)^{m+j+1}q^{j(3j-1)/2-m(15m+11)/2-1}(1+q^j), \end{aligned}$$

$$
\begin{aligned}
&\quad - \sum_{m=-\infty}^{\infty} q^{6m^2+m} + \sum_{m=-\infty}^{\infty} q^{6m^2+5m+1} \\
&= \sum_{j=1}^{\infty} \sum_{-j+1 \le 3m \le j-1} \operatorname{sgn}(m)(-1)^{m+j+1} q^{j(3j-1)/2 - m(15m+1)/2} \\
&\quad + \sum_{j=0}^{\infty} \sum_{-j \le 3m \le j} \operatorname{sgn}(m)(-1)^{m+j+1} q^{j(3j+1)/2 - m(15m+1)/2} \\
&\quad + \sum_{j=2}^{\infty} \sum_{-j \le 3m \le j-2} \operatorname{sgn}(m)(-1)^{m+j+1} q^{j(3j-1)/2 - m(15m+11)/2 - 1} \\
&\quad + \sum_{j=1}^{\infty} \sum_{-j-1 \le 3m \le j-1} \operatorname{sgn}(m)(-1)^{m+j+1} q^{j(3j+1)/2 - m(15m+11)/2 - 1}.
\end{aligned}
$$

On the right side of the last equation above replace j by $j+1$ in the first and third double sums to obtain

$$
\begin{aligned}
&(q)_\infty(\chi_0(q) - 2) \\
&= \sum_{j=0}^{\infty} \sum_{-j \le 3m \le j} \operatorname{sgn}(m)(-1)^{m+j+1} q^{j(3j+1)/2 - m(15m+1)/2}(1 - q^{2j+1}) \\
&+ \sum_{j=1}^{\infty} \sum_{-j-1 \le 3m \le j-1} \operatorname{sgn}(m)(-1)^{m+j+1} q^{j(3j+1)/2 - m(15m+11)/2 - 1}(1 - q^{2j+1}),
\end{aligned}
$$

which is (1.1).

To prove (1.2) we need Slater's [8, p.463] A(2) Bailey pair relative to (q, q):

$$
\beta_n = \frac{1}{(q^2;q)_{2n}}, \qquad \alpha_n = \begin{cases} q^{6m^2-m} & \text{if } n = 3m-1, \\ q^{6m^2+m} & \text{if } n = 3m, \\ -q^{6m^2+5m+1} - q^{6m^2+7m+2} & \text{if } n = 3m+1. \end{cases} \tag{3.4}
$$

We have

$$
\begin{aligned}
\chi_1(q) &= \sum_{n=0}^{\infty} \frac{q^n}{(q^{n+1};q)_{n+1}} = \sum_{n=0}^{\infty} \frac{q^n (q)_n}{(q)_{2n+1}} \\
&= \sum_{n=0}^{\infty} \frac{1}{(q^2;q)_{2n}} \cdot \frac{q^n (q)_n}{(1-q)} = \frac{1}{(q)_\infty} \sum_{n=0}^{\infty} \beta_n \delta_n.
\end{aligned}
$$

By the Bailey Transform and (3.4) we have

(3.5)

$$
\begin{aligned}
&(q)_\infty \chi_1(q) = \sum_{n=0}^{\infty} \beta_n \delta_n = \sum_{n=0}^{\infty} \alpha_n \gamma_n \\
&= \sum_{m=1}^{\infty} \sum_{j=3m}^{\infty} (-1)^{m+j} q^{j(3j-1)/2 - m(15m-7)/2 - 1}(1 + q^j) \\
&\quad + \sum_{m=0}^{\infty} \sum_{j=3m+1}^{\infty} (-1)^{m+j+1} q^{j(3j-1)/2 - m(15m+7)/2 - 1}(1 + q^j)
\end{aligned}
$$

$$+\sum_{m=0}^{\infty}\sum_{j=3m+2}^{\infty}(-1)^{m+j+1}\Big\{q^{j(3j-1)/2-m(15m+17)/2-3}$$
$$+q^{j(3j-1)/2-m(15m+13)/2-2}\Big\}(1+q^j)$$
$$=\sum_{j=1}^{\infty}\sum_{-j\le 3m\le j-1}\operatorname{sgn}(m)(-1)^{m+j+1}q^{j(3j-1)/2-m(15m+7)/2-1}(1+q^j)$$
$$+\sum_{j=1}^{\infty}\sum_{-j-1\le 3m\le j-2}\operatorname{sgn}(m)(-1)^{m+j+1}q^{j(3j-1)/2-m(15m+13)/2-2}(1+q^j),$$

by noting that

$$-(-m-1)(15(-m-1)+17)/2-3=-m(15m+13)/2-2.$$

This completes the proof of Theorem 1.1.

4. Proof of Theorem 1.2

To prove Theorem 1.2 we proceed as in Section 3. This time we need Slater's Bailey pairs $A(6)$ and $A(8)$, and a variant of her Bailey pair $A(7)$.

From [8, Eq.(3.4),p.464] we have

$$\frac{q^{n^2-n}}{(q)_{2n}}=\sum_{r=-[(n+1)/3]}^{[n/3]}\frac{(1-q^{6r+1})q^{3r^2-2r}}{(q)_{n+3r+1}(q)_{n-3r}}$$
$$=\sum_{r=0}^{[n/3]}\frac{(1-q^{6r+1})q^{3r^2-2r}}{(q)_{n-3r}(q)_{n+3r+1}}+\sum_{r=1}^{[(n+1)/3]}\frac{(1-q^{-6r+1})q^{3r^2+2r}}{(q)_{n+3r}(q)_{n+1-3r}}$$
$$=\sum_{r=1}^{[(n+1)/3]}\frac{q^{3r^2+2r}-q^{3r^2-4r+1}}{(q)_{n-(3r-1)}(q)_{n+(3r-1)+1}}+\sum_{r=0}^{[n/3]}\frac{q^{3r^2-2r}-q^{3r^2+4r+1}}{(q)_{n-3r}(q)_{n+3r+1}},$$

so that

$$\frac{(1-q)q^{n^2-n}}{(q)_{2n}}$$
$$=\sum_{r=1}^{[(n+1)/3]}\frac{q^{3r^2+2r}-q^{3r^2-4r+1}}{(q)_{n-(3r-1)}\,(q^2;q)_{n+(3r-1)}}+\sum_{r=0}^{[n/3]}\frac{q^{3r^2-2r}-q^{3r^2+4r+1}}{(q)_{n-3r}\,(q^2;q)_{n+3r}}.$$

This implies the following Bailey pair relative to (q,q):

$$\beta_n=\frac{(1-q)q^{n^2-n}}{(q)_{2n}},\qquad \alpha_n=\begin{cases} q^{3m^2+2m}-q^{3m^2-4m+1} & \text{if } n=3m-1,\\ q^{3m^2-2m}-q^{3m^2+4m+1} & \text{if } n=3m,\\ 0 & \text{if } n=3m+1.\end{cases}\tag{4.1}$$

We note that this Bailey pair was found by Warnaar [10, p.375] using a different method. We have

$$\mathcal{F}_0(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q^{n+1};q)_n}$$
$$=\sum_{n=0}^{\infty}\frac{(1-q)q^{n^2-n}}{(q)_{2n}}\cdot\frac{q^n(q)_n}{(1-q)}=\frac{1}{(q)_\infty}\sum_{n=0}^{\infty}\beta_n\delta_n.$$

Thus by the Bailey Transform and (4.1) we have

$$
\begin{aligned}
(q)_\infty \mathcal{F}_0(q) &= \sum_{n=0}^{\infty} \beta_n \delta_n = \sum_{n=0}^{\infty} \alpha_n \gamma_n \\
&= \sum_{m=1}^{\infty} \sum_{j=3m}^{\infty} (-1)^{m+j} q^{j(3j-1)/2 - m(21m-13)/2-1}(1+q^j) \\
&\quad + \sum_{m=1}^{\infty} \sum_{j=3m}^{\infty} (-1)^{m+j+1} q^{j(3j-1)/2 - m(21m-1)/2}(1+q^j) \\
&\quad + \sum_{m=0}^{\infty} \sum_{j=3m+1}^{\infty} (-1)^{m+j+1} q^{j(3j-1)/2 - m(21m+13)/2-1}(1+q^j) \\
&\quad + \sum_{m=0}^{\infty} \sum_{j=3m+1}^{\infty} (-1)^{m+j} q^{j(3j-1)/2 - m(21m+1)/2}(1+q^j) \\
&= \sum_{j=1}^{\infty} \sum_{-j \le 3m \le j-1} \operatorname{sgn}(m)(-1)^{m+j+1} q^{j(3j-1)/2 - m(21m+13)/2-1}(1+q^j) \\
&\quad + \sum_{j=1}^{\infty} \sum_{-j \le 3m \le j-1} \operatorname{sgn}(m)(-1)^{m+j} q^{j(3j-1)/2 - m(21m+1)/2}(1+q^j), \\
&= \sum_{j=1}^{\infty} \sum_{-j \le 3m \le j-1} \operatorname{sgn}(m)(-1)^{m+j+1} q^{j(3j-1)/2 - m(21m+13)/2-1} \\
&\qquad\qquad (1+q^j)(1-q^{6m+1}),
\end{aligned}
\tag{4.2}
$$

which is (1.3).

To prove (1.4) we need Slater's [8, p.463] A(8) Bailey pair relative to (q, q):

$$
\beta_n = \frac{q^{n^2+n}}{(q^2;q)_{2n}}, \qquad \alpha_n = \begin{cases} q^{3m^2-2m} & \text{if } n = 3m-1, \\ q^{3m^2+2m} & \text{if } n = 3m, \\ -q^{3m^2+4m+1} - q^{3m^2+2m} & \text{if } n = 3m+1. \end{cases}
\tag{4.3}
$$

We have

$$
\begin{aligned}
\mathcal{F}_1(q) &= \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q^n;q)_n} = q \sum_{n=0}^{\infty} \frac{q^{n^2+2n}}{(q^{n+1};q)_{n+1}} \\
&= q \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q^2;q)_{2n}} \cdot \frac{q^n (q)_n}{(1-q)} = \frac{q}{(q)_\infty} \sum_{n=0}^{\infty} \beta_n \delta_n.
\end{aligned}
$$

Thus by the Bailey Transform and (4.3) we have

(4.4)

$$
\begin{aligned}
(q)_\infty \mathcal{F}_1(q) &= \sum_{n=0}^{\infty} \beta_n \delta_n = \sum_{n=0}^{\infty} \alpha_n \gamma_n \\
&= \sum_{m=1}^{\infty} \sum_{j=3m}^{\infty} (-1)^{m+j} q^{j(3j-1)/2 - m(21m-5)/2}(1+q^j)
\end{aligned}
$$

$$+\sum_{m=0}^{\infty}\sum_{j=3m+1}^{\infty}(-1)^{m+j+1}q^{j(3j-1)/2-m(21m+5)/2}(1+q^j)$$
$$+\sum_{m=0}^{\infty}\sum_{j=3m+2}^{\infty}(-1)^{m+j+1}\left\{q^{j(3j-1)/2-m(21m+19)/2-2}(1+q^j)\right.$$
$$\left.+q^{j(3j-1)/2-m(21m+23)/2-3}(1+q^j)\right\}$$
$$=\sum_{j=1}^{\infty}\sum_{-j\le 3m\le j-1}\operatorname{sgn}(m)(-1)^{m+j+1}q^{j(3j-1)/2-m(21m+5)/2}(1+q^j)$$
$$+\sum_{j=2}^{\infty}\sum_{-j-1\le 3m\le j-2}\operatorname{sgn}(m)(-1)^{m+j+1}q^{j(3j-1)/2-m(21m+19)/2-2}(1+q^j),$$

which is (1.4).

To prove (1.5) we need Slater's [8, p.463] A(6) Bailey pair relative to (q,q):

$$\beta_n=\frac{q^{n^2}}{(q^2;q)_{2n}},\qquad \alpha_n=\begin{cases} q^{3m^2+m} & \text{if } n=3m-1,\\ q^{3m^2-m} & \text{if } n=3m,\\ -q^{3m^2+m}-q^{3m^2+5m+2} & \text{if } n=3m+1,\end{cases}\tag{4.5}$$

We have

$$\mathcal{F}_2(q)=\sum_{n=1}^{\infty}\frac{q^{n^2+n}}{(q^{n+1};q)_{n+1}}=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q^2;q)_{2n}}\cdot\frac{q^n(q)_n}{(1-q)}=\frac{1}{(q)_\infty}\sum_{n=0}^{\infty}\beta_n\delta_n.$$

Thus by the Bailey Transform and (4.5) we have

(4.6)

$$(q)_\infty\mathcal{F}_2(q)=\sum_{n=0}^{\infty}\beta_n\delta_n=\sum_{n=0}^{\infty}\alpha_n\gamma_n$$
$$=\sum_{m=1}^{\infty}\sum_{j=3m}^{\infty}(-1)^{m+j}q^{j(3j-1)/2-m(21m-11)/2-1}(1+q^j)$$
$$+\sum_{m=0}^{\infty}\sum_{j=3m+1}^{\infty}(-1)^{m+j+1}q^{j(3j-1)/2-m(21m+11)/2-1}(1+q^j)$$
$$+\sum_{m=0}^{\infty}\sum_{j=3m+2}^{\infty}(-1)^{m+j+1}\left\{q^{j(3j-1)/2-m(21m+25)/2-4}(1+q^j)\right.$$
$$\left.+q^{j(3j-1)/2-m(21m+17)/2-2}(1+q^j)\right\}$$
$$=\sum_{j=1}^{\infty}\sum_{-j\le 3m\le j-1}\operatorname{sgn}(m)(-1)^{m+j+1}q^{j(3j-1)/2-m(21m+11)/2-1}(1+q^j)$$
$$+\sum_{j=2}^{\infty}\sum_{-j-1\le 3m\le j-2}\operatorname{sgn}(m)(-1)^{m+j+1}q^{j(3j-1)/2-m(21m+17)/2-2}(1+q^j),$$

which is (1.5). This completes the proof of Theorem 1.2.

5. Zagier's Mock Theta Function Identities and Related Results

In this section we write our double-series identities for the two fifth order functions $\chi_0(q)$ and $\chi_1(q)$ and all three seventh order functions $\mathcal{F}_j(q)$ $(j = 0, 1, 2)$ using Dirichlet characters. This leads naturally to relations between the coefficients of these series as in Theorems 5.5 and 5.6.

As mentioned before Andrews [1] obtained indefinite theta series identities for all of Ramanujan's fifth order functions except $\chi_0(q)$ and $\chi_1(q)$. Using Andrews's results Zwegers [14] showed how to complete all of Andrews's fifth order functions to weak harmonic Maass forms. As noted by Watson [11, pp.277-279], Ramanujan gave identities for $\chi_0(q)$ and $\chi_1(q)$ in terms of the other fifth order functions. Zagier suggested that indefinite theta function identities for $\chi_0(q)$ and $\chi_1(q)$ could be obtained from Ramanujan's results and Zwegers transformation formulas, although he gave no details. We state Zagier's results in a modified form in the following

Theorem 5.1.

$$(q)_\infty(2-\chi_0(q)) = \sum_{\substack{5|b|<|a| \\ a+b\equiv 2 \pmod 4 \\ a\equiv 2 \pmod 5}} (-1)^a \operatorname{sgn}(a)\left(\frac{-3}{a^2-b^2}\right) q^{\frac{1}{120}(a^2-5b^2)-\frac{1}{30}}$$

and

$$(q)_\infty \chi_1(q) = \sum_{\substack{5|b|<|a| \\ a+b\equiv 2 \pmod 4 \\ a\equiv 4 \pmod 5}} (-1)^a \operatorname{sgn}(a)\left(\frac{-3}{a^2-b^2}\right) q^{\frac{1}{120}(a^2-5b^2)-\frac{19}{30}}$$

Remark 5.2. Here $\left(\frac{-3}{\cdot}\right)$ is the Kronecker symbol, and is a Dirichlet character mod 3.

Our Theorem 1.1 seems to differ from Zagier's Theorem. In contrast to Zagier's theorem which involves a character mod 3 our version involves the Dirichlet character mod 60:

$$\chi_{60}(m) = \begin{cases} 1 & \text{if } m \equiv 1,\ 11,\ 19,\ 29 \pmod{60} \\ i & \text{if } m \equiv 7,\ 13,\ 17,\ 23 \pmod{60} \\ -1 & \text{if } m \equiv 31,\ 41,\ 49,\ 59 \pmod{60} \\ -i & \text{if } m \equiv 37,\ 43,\ 47,\ 53 \pmod{60} \end{cases}$$

Theorem 5.3.

$$(q)_\infty(2-\chi_0(q)) = \sum_{\substack{3|b|<5|a| \\ a\equiv 1 \pmod 6 \\ b\equiv 1,11 \pmod{30}}} \operatorname{sgn}(b)\left(\frac{12}{a}\right)\chi_{60}(b) q^{\frac{1}{120}(5a^2-b^2)-\frac{1}{30}} \tag{5.1}$$

and

$$(q)_\infty \chi_1(q) = i \sum_{\substack{3|b|<5|a| \\ a\equiv b\equiv 1 \pmod 6 \\ b\equiv \pm 2 \pmod 5}} \operatorname{sgn}(b)\left(\frac{12}{a}\right)\chi_{60}(b) q^{\frac{1}{120}(5a^2-b^2)-\frac{19}{30}} \tag{5.2}$$

We find analogous identities for the seventh order functions. Also Andrews [1] obtained indefinite theta series identities for these functions. Hickerson [6, Theorem 2.0,p.666] found nice versions of Andrews identities, which he used to prove his seventh order analogues of Ramanujan's mock theta conjectures [3] for the fifth order functions. Our identities differ from Andrews's and Hickerson's and appear to be new.

Theorem 5.4.

$$(q)_\infty \mathcal{F}_0(q) = \sum_{\substack{3|b|<7|a| \\ a\equiv 1 \pmod 6 \\ b\equiv 1,13 \pmod{42}}} \operatorname{sgn}(b) \left(\frac{12}{a}\right)\left(\frac{12}{b}\right)\left(\frac{b}{7}\right) q^{\frac{1}{168}(7a^2-b^2)-\frac{1}{28}}, \tag{5.3}$$

$$(q)_\infty \mathcal{F}_1(q) = -\sum_{\substack{3|b|<7|a| \\ a\equiv 1 \pmod 6 \\ b\equiv 5,19 \pmod{42}}} \operatorname{sgn}(b) \left(\frac{12}{a}\right)\left(\frac{12}{b}\right)\left(\frac{b}{7}\right) q^{\frac{1}{168}(7a^2-b^2)+\frac{3}{28}}, \tag{5.4}$$

and

$$(q)_\infty \mathcal{F}_2(q) = -\sum_{\substack{3|b|<7|a| \\ a\equiv 1 \pmod 6 \\ b\equiv 11,17 \pmod{42}}} \operatorname{sgn}(b) \left(\frac{12}{a}\right)\left(\frac{12}{b}\right)\left(\frac{b}{7}\right) q^{\frac{1}{168}(7a^2-b^2)-\frac{9}{28}}. \tag{5.5}$$

We sketch the proof of (5.2). Firstly we observe that

$$\begin{aligned}
\tfrac{1}{2}j(3j\pm 1) - \tfrac{1}{2}m(15m+7) - 1 &= \tfrac{1}{120}\left(5(6j\pm 1)^2 - (30m+7)^2\right) - \tfrac{19}{30}, \\
\tfrac{1}{2}j(3j\pm 1) - \tfrac{1}{2}m(15m+13) - 2 &= \tfrac{1}{120}\left(5(6j\pm 1)^2 - (30m+13)^2\right) - \tfrac{19}{30}.
\end{aligned}$$

In the summations in equation (5.2), we let $a = 6(\pm j)+1$, and $b = 30m+r$, where $j \ge 1$, $m \in \mathbb{Z}$, and $r = 7$, 13. We have

$$\begin{aligned}
\left(\frac{12}{a}\right) &= \left(\frac{12}{6(\pm j)+1}\right) = (-1)^j, \\
i\chi_{60}(b) &= i\chi_{60}(30m+r) = (-1)^{m+1}, \quad \text{and} \\
\operatorname{sgn}(b) &= \operatorname{sgn}(30m+r) = \operatorname{sgn}(m).
\end{aligned}$$

Next we consider the inequalities for the variables in the summations.

Case 1. $m \ge 0$ and $r = 7$. Then we see that

$$3|b| < 5|a| \Leftrightarrow 3m < j + \left(\frac{\pm 5 - 21}{30}\right) \Leftrightarrow 3m \le j-1.$$

Case 2. $m < 0$ and $r = 7$. Then we see that

$$3|b| < 5|a| \Leftrightarrow -j < 3m + \left(\frac{\pm 5 + 21}{30}\right) \Leftrightarrow -j \le 3m.$$

Case 3. $m \ge 0$ and $r = 13$. Then we see that

$$3|b| < 5|a| \Leftrightarrow 3m < j + \left(\frac{\pm 5 - 39}{30}\right) \Leftrightarrow 3m \le j-2.$$

Case 4. $m < 0$ and $r = 13$. Then we see that

$$3|b| < 5|a| \Leftrightarrow -j + \left(\frac{-39 \pm 5}{30}\right) < 3m \Leftrightarrow -j - 1 \le 3m.$$

It follows that

$$\sum_{j=1}^{\infty} \sum_{-j \le 3m \le j-1} \operatorname{sgn}(m)(-1)^{m+j+1} q^{j(3j-1)/2 - m(15m+7)/2 - 1}(1 + q^j)$$
$$= i \sum_{\substack{3|b|<5|a| \\ a\equiv 1 \pmod 6 \\ b \equiv 7 \pmod{30}}} \operatorname{sgn}(b) \left(\frac{12}{a}\right) \chi_{60}(b) q^{\frac{1}{120}(5a^2 - b^2) - \frac{19}{30}},$$

and

$$\sum_{j=1}^{\infty} \sum_{-j-1 \le 3m \le j-2} \operatorname{sgn}(m)(-1)^{m+j+1} q^{j(3j-1)/2 - m(15m+13)/2 - 2}(1 + q^j)$$
$$= i \sum_{\substack{3|b|<5|a| \\ a\equiv 1 \pmod 6 \\ b \equiv 13 \pmod{30}}} \operatorname{sgn}(b) \left(\frac{12}{a}\right) \chi_{60}(b) q^{\frac{1}{120}(5a^2 - b^2) - \frac{19}{30}}.$$

Therefore we see that equation (1.2) implies (5.2). The proof of the remaining parts of Theorems 5.3 and 5.4 are analogous.

Theorems 5.3 and 5.4 imply simple relations between the coefficients. We define the coefficients $C_0(n)$ and $C_1(n)$ by

$$\sum_{n=0}^{\infty} C_0(n) q^n = (q)_\infty \, (2 - \chi_0(q)),$$
$$\sum_{n=0}^{\infty} C_1(n) q^n = (q)_\infty \, \chi_1(q),$$

define

$$\varepsilon_p = \begin{cases} -1 & \text{if } p \equiv 3 \pmod{10}, \\ 1 & \text{if } p \equiv 7 \pmod{10}, \end{cases} \tag{5.6}$$

and for an integer n and a prime p, define $\nu_p(n)$ to be the exact power of p dividing n.

Theorem 5.5. *If $p > 5$ is any prime congruent to 3 or 7 mod 10, then*

$$C_0(n) = 0 \qquad \text{if } \nu_p(30n+1) = 1, \tag{5.7}$$
$$C_0(p^2 n + \tfrac{1}{30}(19p^2 - 1)) = -\varepsilon_p \, C_1(n) \qquad \text{for } n \ge 0, \tag{5.8}$$
$$C_1(n) = 0 \qquad \text{if } \nu_p(30n+19) = 1, \tag{5.9}$$
$$C_1(p^2 n + \tfrac{1}{30}(p^2 - 19)) = \varepsilon_p \, C_0(n) \qquad \text{for } n \ge 0. \tag{5.10}$$

Proof. Suppose $p > 5$ is any prime congruent to 3 or 7 mod 10. Then 5 is a quadratic nonresidue mod p. Therefore $5a^2 - b^2 \equiv 0 \pmod p$ implies that $a \equiv b \equiv 0 \pmod p$ and (5.7) clearly follows from (5.1). Similarly (5.9) follows from (5.2).

We suppose $a \equiv 1 \pmod 6$, $b \equiv 1, 11 \pmod{30}$, $3|b| < 5|a|$, and $a \equiv b \equiv 0 \pmod p$. Letting $a = pa'$, $b = pb'$ we have the following table

$p \pmod{30}$	$a' \pmod 6$	$b' \pmod{30}$
7	1	$13, -7$
13	1	$7, -13$
17	-1	$-7, 13$
23	-1	$-13, 7$

By considering the table and noting that the summation term

$$\operatorname{sgn}(b)\left(\frac{12}{a}\right)\chi_{60}(b)q^{\frac{1}{120}(5a^2-b^2)-\frac{1}{30}}$$

is invariant under both $a \mapsto -a$ and $b \mapsto -b$ we see that

$$\begin{aligned}
&\sum_{n=0}^{\infty} C_0(p^2n + \tfrac{1}{30}(19p^2-1))\, q^{p^2n+\frac{1}{30}(19p^2-1)} \\
&= \sum_{\substack{3|b'|<5|a'| \\ a'\equiv 1 \pmod 6 \\ b'\equiv 7,13 \pmod{30}}} \operatorname{sgn}(pb')\left(\frac{12}{pa'}\right)\chi_{60}(pb')q^{\frac{1}{120}(p^2(5(a')^2-(b')^2)-\frac{1}{30}} \\
&= \left(\frac{12}{p}\right)\chi_{60}(p) \sum_{\substack{3|b|<5|a| \\ a\equiv 1 \pmod 6 \\ b\equiv 7,13 \pmod{30}}} \operatorname{sgn}(b)\left(\frac{12}{a}\right)\chi_{60}(b)q^{\frac{1}{120}(p^2(5a^2-b^2))-\frac{1}{30}}
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{n=0}^{\infty} C_0(p^2n + \tfrac{1}{30}(19p^2-1))\, q^{n} \\
&= -i\,\varepsilon_p \sum_{\substack{3|b|<5|a| \\ a\equiv 1 \pmod 6 \\ b\equiv 7,13 \pmod{30}}} \operatorname{sgn}(b)\left(\frac{12}{a}\right)\chi_{60}(b)q^{\frac{1}{120}((5a^2-b^2)-\frac{19}{30}} \\
&= -\varepsilon_p(q)_\infty\, \chi_1(q) = -\varepsilon_p \sum_{n=0}^{\infty} C_1(n)q^n,
\end{aligned}$$

and (5.8) follows. The proof of (5.9)–(5.10) is analogous. ☐

In a similar fashion, Theorem 5.4 implies relations between the coefficients of the seventh order mock theta functions. For $j = 0, 1, 2$ we define $f_j(n)$ by

$$\sum_{n=0}^{\infty} f_j(n)q^n = (q)_\infty\, \mathcal{F}_j(q).$$

Theorem 5.6. *Let p be any odd for which 7 is a quadratic nonresidue mod p;i.e. $p \equiv \pm 5, \pm 11$ or $\pm 13 \pmod{28}$.*

(1) *Then*

$$\begin{aligned}
f_0(n) &= 0 \qquad \text{if } \nu_p(28n+1) = 1, \\
f_1(n) &= 0 \qquad \text{if } \nu_p(28n-3) = 1, \\
f_2(n) &= 0 \qquad \text{if } \nu_p(28n+9) = 1.
\end{aligned}$$

(2) *If* $p \equiv \pm 5 \pmod{28}$ *then*

$$f_0(p^2n + \tfrac{1}{28}(9p^2-1)) = \pm f_2(n),$$
$$f_1(p^2n + \tfrac{1}{28}(p^2+3)) = \pm f_0(n),$$
$$f_2(p^2n + \tfrac{1}{28}(25p^2-9)) = \mp f_1(n+1).$$

(3) *If* $p \equiv \pm 11 \pmod{28}$ *then*

$$f_0(p^2n + \tfrac{1}{28}(25p^2-1)) = \mp f_1(n+1),$$
$$f_1(p^2n + \tfrac{1}{28}(9p^2+3)) = \pm f_2(n),$$
$$f_2(p^2n + \tfrac{1}{28}(p^2-9)) = \mp f_0(n).$$

(4) *If* $p \equiv \pm 13 \pmod{28}$ *then*

$$f_0(p^2n + \tfrac{1}{28}(p^2-1)) = \mp f_0(n),$$
$$f_1(p^2n + \tfrac{1}{28}(25p^2+3)) = \mp f_1(n+1),$$
$$f_2(p^2n + \tfrac{1}{28}(9p^2-9)) = \mp f_2(n).$$

We omit the proof of Theorem 5.6. The proof is analogous to that of Theorem 5.5.

6. Concluding Remarks

In Theorems 5.3 and 5.4 we found new identities for the fifth order mock theta functions $\chi_0(q)$, $\chi_1(q)$ and all three seventh order mock theta functions $\mathcal{F}_0(q)$, $\mathcal{F}_1(q)$, $\mathcal{F}_2(q)$, in terms of Hecke-Rogers indefinite binary theta series. This suggests the problem of relating these theorems directly to the results of Zagier (Theorem 5.1) for the fifth order functions, and to the results of Andrews [1, Theorem 13, pp.132–133] and Hickerson [6, Theorem 2.0,p.666] for the seventh order functions.

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