q-SERIES AND PARTITIONS

References

① George E. Andrews, The Theory of Partitions, Cambridge Univ. Press, 1998

② G. Gasper and M. Rahman, Basic Hypergeometric Series, Cambridge Univ. Press, 1990

③ G.E. Andrews, R. Askey & R. Roy, Special Functions, Cambridge Univ Press, 1999.

ADDITIVE NUMBER THEORY

Basic Problem Let $A = \{a_1, a_2, \ldots\}$ be a set of positive integers. For which $n \geq 1$ can n be written as a sum of integers from A. Let $A(n)$ = number of such representations.

Goldbach's Conjecture (1742) Every even $n > 4$ is the sum of two odd primes.

Example $6 = 3+3$, $8 = 5+3$, $10 = 7+3 = 5+5$.

Sums of squares Let $k \geq 2$. Let $r_k(n)$ be number of solutions to

$$n = x_1^2 + x_2^2 + \cdots + x_k^2$$

where each $x_i \in \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$ (set of integers), and order matters. Jacobi (1829) found simple exact formulas for $r_k(n)$ where $k = 2, 4, 6$ or 8. Example:

$$r_2(n) = 4\,(d_1(n) - d_3(n))$$

where $d_j(n)$ = # of (positive) divisors of n congruent to j (mod 4).

(2)

Example $n=5$.

$5 = 2^2+1^2 = (-2)^2+1^2 = 2^2+(-1)^2 = (-2)^2+(-1)^2$
$= 1^2+2^2 = 1^2+(-2)^2 = (-1)^2+2^2 = (-1)^2+(-2)^2$

So $r_2(5) = 8$.

$d_1(5) = 2$ since $1, 5 \mid 5$.

$d_3(5) = 0$.

Waring's Problem Let $k \geq 1$ be given. Determine if there is a integer s (depending only on k) such that the equation

$$n = x_1^k + x_2^k + \cdots + x_s^k$$

has solutions for every $n \geq 1$.

Example

Every $n \geq 1$ be be represented as a sum of 4 squares.

. . . . 9 cubes.

. . . . 19 fourth powers.

Let $g(k)$ be the least value of s.

$g(2) = 4$ [Lagrange (1770)]

$g(3) = 9$ Wieferich (1909)

$g(4) = 19$ Balasubramanian, Deshouillers, Dress (1986)

$g(5) = 37$ Chen (1964).

Unrestricted partitions Let $n \geq 1$. An (unrestricted) partition of n is a representation of n as a sum of positive integers

$$n = a_1 + a_2 + \cdots + a_k.$$

Order of parts is irrelevant and k is not restricted.

Usually the parts are arranged in descending order.

(3)

Let $p(n)$ denote the number of partitions of n.

Example

n	partitions of n	$p(n)$
1	1	1
2	2, 1+1	2
3	3, 2+1, 1+1+1	3
4	4, 3+1, 2+2, 2+1+1 1+1+1+1	5
5	5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1, 1+1+1+1+1	7
6	6, 5+1, ... , 1+1+1+1+1+1	11
7	7, 6+1, ... , 1+1+1+1+1+1+1	15
100		190569292

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{2n/3}}$$ Hardy & Ramanujan (1918)

as $n \to \infty$.

Note We say $f(n) \sim g(n)$ as $n \to \infty$ if
$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1.$$

We will prove the following (4)

$$\sum_{n=0}^{\infty} p(n) q^n = 1 + p(1) q^1 + p(2) q^2 + p(3) q^3 + p(4) q^4 + \cdots$$
$$= 1 + q + 2q^2 + 3q^3 + 5q^4 + \cdots$$
$$= \prod_{m=1}^{\infty} \frac{1}{1-q^m} = \frac{1}{(1-q)} \cdot \frac{1}{(1-q^2)} \cdot \frac{1}{(1-q^3)} \cdots$$

for $|q| < 1$ (Euler) (Convention: $p(0)=1$).

For $n \geq 1$,

$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7)$$
$$+ p(n-12) + p(n-15)$$
$$+ \cdots + (-1)^{k+1}\left(p\left(n - \frac{k(3k-1)}{2}\right) + p\left(n - \frac{k(3k+1)}{2}\right)\right)$$
$$+ \cdots$$

(Euler) (Convention: $p(m)=0$ if $m<0$).

Example:

$$p(6) = p(6-1) + p(6-2) - p(6-5)$$
$$= p(5) + p(4) - p(1) = 7+5-1 = 11$$
$$p(7) = p(7-1) + p(7-2) - p(7-5) - p(7-7)$$
$$= p(6) + p(5) - p(2) - p(0)$$
$$= 11 + 7 - 2 - 1$$
$$= 15.$$

Convergence of Infinite Products

Defn $\prod_{n=1}^{\infty}(1+a_n)$ converges if

$\lim_{N\to\infty} \prod_{n=1}^{N}(1+a_n)$ exist and does not equal zero. We say the product converges absolutely if $\prod_{n=1}^{\infty}(1+|a_n|)$ converges.

<u>Theorem</u> Suppose $a_n > -1$ for all n (or more generally $\mathrm{Re}\, a_n > -1$ for all n). The product $\prod_{n=1}^{\infty}(1+a_n)$ converges iff $\sum_{n=1}^{\infty} \log(1+a_n)$ converges.

<u>Theorem</u> The product $\prod_{n=1}^{\infty}(1+a_n)$ converges absolutely iff $\sum_{n=1}^{\infty} a_n$ converges absolutely.

<u>Example</u> $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges absolutely. So

$$\prod_{n=1}^{\infty}\left(1+\frac{1}{n^2}\right) \text{ converges absolutely.}$$

In fact,

$$\prod_{n=1}^{\infty}\left(1+\frac{1}{n^2}\right) = \frac{e^{\pi}+e^{-\pi}}{2\pi}.$$

<u>Theorem</u> (Euler) For $|q|<1$,

$$\sum_{n=0}^{\infty} p(n)\, q^n = \cancel{\prod_{n=1}^{\infty}\frac{1}{1-q^n}} \cdot \prod_{k=1}^{\infty}\frac{1}{1-q^k}.$$

<u>~~Proof~~ Discussion</u>.

For $|q|<1$,

$$\frac{1}{1-q} = \sum_{n=0}^{\infty} q^n = 1+q+q^2+q^3+\cdots$$

$$\frac{1}{1-q^2} = \sum_{n=0}^{\infty} q^{2n} = 1+q^2+q^4+q^6+\cdots$$

$$\frac{1}{1-q^k} = \sum_{n=0}^{\infty} q^{kn} = 1+q^k+q^{2k}+q^{3k}+\cdots$$

(6)

$$\frac{1}{1-q}\cdot\frac{1}{1-q^2}\cdots\frac{1}{1-q^m}$$

$$= (1+q^1+q^{1+1}+q^{1+1+1}+\cdots)(1+q^2+q^{2+2}+q^{2+2+2}+\cdots)$$
$$(1+q^3+q^{3+3}+q^{3+3+3}+\cdots)\cdots\cdots(1+q^m+q^{m+m}+\cdots)$$

$$= \sum_{n_1=0}^{\infty} q^{n_1}\sum_{n_2=0}^{\infty} q^{2n_2}\cdots\sum_{n_m=0}^{\infty} q^{n_m m} = \sum q^{1\cdot n_1+2n_2+\cdots+m n_m}$$

$$= 1 + q^1 + (q^{1+1}+q^2) + (q^{1+1+1}+q^{1+2}+q^3)+\cdots$$

$$= 1 + \sum_{n=1}^{\infty} p_m(n)\, q^n$$

where $p_m(n)$ = # of solutions of
$$n = a_1+a_2+\cdots+a_k$$
where $m \geq a_1 \geq a_2 \geq \cdots \geq a_k \geq 1$, $1 \leq k \leq m$

= # of partitions of n into ~~at most~~ ~~m~~ parts not exceeding m.

<u>Theorem</u>: Let $m \geq 1$. Then for $|q|<1$

$$\sum_{n=0}^{\infty} p_m(n)\, q^n = \left(\frac{1}{1-q}\right)\left(\frac{1}{1-q^2}\right)\cdots\left(\frac{1}{1-q^m}\right)$$
$$= \prod_{k=1}^{m}\frac{1}{1-q^k}$$

and $p_m(n)$ = # of partitions of n into ~~at most~~ ~~m~~ parts not exceeding m.

<u>Example</u> Find $p_3(6)$.

$$\sum_{n=0}^{\infty} p_3(n)\, q^n = \left(\frac{1}{1-q}\right)\left(\frac{1}{1-q^2}\right)\left(\frac{1}{1-q^3}\right).$$

MAPLE: (7)

> x := 1/(1-q)/(1-q^2)/(1-q^3);

$$x := \frac{1}{(1-q)(1-q^2)(1-q^3)}$$

> series(x, q, 10);

$$1 + q + 2q^2 + 3q^3 + 4q^4 + 5q^5 + 7q^6 + 8q^7 + 10q^8 + 12q^9 + O(q^{10}).$$

So $p_3(6) = 7$.

Partitions of 6 into parts ≤ 3:

3 + 3	2 + 2 + 2	1+1+1+1+1+1
3 + 2 + 1	2+2+1+1	
3 + 1+1+1	2+1+1+1+1	

<u>Theorem</u>

Let $p_{m,d}(n)$ = # of partitions of n into parts $\leq m$ and each part occurs at most d times

Then

$$\sum_{n=0}^{\infty} p_{m,d}(n)\, q^n = (1+q^1+q^{1+1}+\cdots+q^d)$$

$$(1+q^2+q^{2+2}+\cdots\cdots+q^{2d})$$

$$\vdots$$

$$(1+q^m+q^{2m}+\cdots\cdots+q^{md})$$

$$= \frac{(1-q^{d+1})}{(1-q)} \frac{(1-q^{2d+2})}{(1-q^2)} \cdots\cdots \frac{(1-q^{md+m})}{(1-q^m)}$$

$$= \prod_{k=1}^{m} \frac{(1-q^{k(d+1)})}{(1-q^k)}$$

<u>note</u>: $1+x+\cdots+x^d = \dfrac{x^{d+1}-1}{x-1}$

Example: Find G.F. for $p_{3,3}(n)$. (8)

$$\sum_{n=0}^{\infty} p_{3,3}(n) q^n = \prod_{k=1}^{3} \frac{(1-q^{4k})}{(1-q^k)}$$

> mul((1-q^(4*k))/((1-q^k), k=1..3);

$$\frac{(1-q^4)(1-q^8)(1-q^{12})}{(1-q)(1-q^2)(1-q^3)}$$

> normal(%);

$$(q^8-q^7+q^6+q^2-q+1)(q^7+q^6+q^5+q^4+q^3+q^2+q+1)(q^3+q^2+q+1)$$

> expand(%);

> sort(%);

$$q^{18}+q^{17}+2q^{16}+3q^{15}+3q^{14}+4q^{13}+5q^{12}+5q^{11}+5q^{10}+6q^9+5q^8+5q^7+5q^6+4q^5+3q^4+3q^3+2q^2+q+1$$

Proof of Eulers Theorem:

Let $m \geq 1$. If $n \leq m$ then any partition

$$n = a_1 + a_2 + \cdots + a_k$$

has $a_k \leq a_{k-1} \leq \cdots \leq a_1 \leq n \leq m$, and

$$p(n) = p_m(n).$$

(9

Let $F_m(q) := \prod_{k=1}^{m} \frac{1}{1-q^k}$, $F(q) = \prod_{k=1}^{\infty} \frac{1}{1-q^k}$

The product $\prod_{k=1}^{\infty}(1-q^k)$ converges absolutely since $\sum_{k=1}^{\infty} q^k$ converges absolutely for $|q|<1$. Hence the product $\prod_{k=1}^{\infty} \frac{1}{1-q^k} = \frac{1}{\prod_{k=1}^{\infty}(1-q^k)}$ converges.

Now suppose $0 \leq q < 1$.

$$F_{m+1}(q) = \frac{1}{1-q^{m+1}} F_m(q) \geq F_m(q),$$

and $\{F_m(q)\}$ is an increasing sequence which converges to $F(q)$.

Hence $0 < F_m(q) \leq F(q)$.

$$F_m(q) = \sum_{n=0}^{\infty} p_m(n) q^n$$

$$= \sum_{n=0}^{m} p_m(n) q^n + \sum_{n=m+1}^{\infty} p_m(n) q^n$$

$$= \sum_{n=0}^{m} p(n) q^n + \sum_{n=m+1}^{\infty} p_m(n) q^n$$

$$\sum_{n=0}^{m} p(n) q^n \leq F_m(q) \leq F(q)$$

Therefore the series $\sum_{n=0}^{\infty} p(n) q^n$ converges, and

$$\sum_{n=0}^{\infty} p(n) q^n \leq F(q)$$

Also,

$$p_m(n) \leq p(n) \quad \text{for all } n$$

(10)

Hence

$$F_m(q) = \sum_{n=0}^{\infty} p_m(n)\, q^n \leq \sum_{n=0}^{\infty} p(n)\, q^n \leq F(q).$$

So

$$\lim_{n\to\infty} F_m(q) = F(q) \leq \sum_{n=0}^{\infty} p(n)\, q^n \leq F(q),$$

and

$$\sum_{n=0}^{\infty} p(n)\, q^n = F(q) = \prod_{k=1}^{\infty} \frac{1}{1-q^k}$$

for $0 \leq q < 1$. The result can be extended to $|q|<1$ by analytic continuation.

Formally, a partition π is a k-tuple

$$\pi = (\lambda_1, \lambda_2, \ldots, \lambda_k) \qquad (\text{some } k \geq 1)$$

of positive integers

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k.$$

π is a partition of n if

$$n = \lambda_1 + \lambda_2 + \cdots + \lambda_k.$$

We write $|\pi| := \lambda_1 + \lambda_2 + \cdots + \lambda_k$ (weight of π or sum of parts). Let $\mathcal{P}$ be the set of partitions.

$$p(n) = \sum_{\substack{\pi \in \mathcal{P} \\ |\pi| = n}} 1,$$

and

$$\sum_{n=0}^{\infty} p(n)\, q^n = \sum_{\pi \in \mathcal{P}} q^{|\pi|}$$

Empty partition $\pi = ()$, $|\pi| = 0$.

Defn:

Let $\mathcal{S} \subset \mathcal{P}$. (11)

$$p(\mathcal{S}, n) := \sum_{\substack{\pi \in \mathcal{S} \\ |\pi| = n}} 1 .$$

so that

$$\sum_{n=0}^{\infty} p(\mathcal{S}, n)\, q^n = \sum_{\pi \in \mathcal{S}} q^{|\pi|} \qquad \text{for } |q|<1.$$

Let $H \subseteq \mathbb{Z}^+$ (set of positive integers).

Let

$$`H' := \{ \pi = (\lambda_1, \ldots, \lambda_k) \in \mathcal{P} : k \geq 1 \ \& \text{ each } \lambda_i \in H \}$$

Thus

$$p(`H', n) = \#\text{ of partitions of } n \text{ whose part are elements of } H.$$

Example: ① Let $\mathcal{O}$ = set of odd positive integers $= \{1, 3, 5, 7, \ldots\}$.

Let $O = `\mathcal{O}'$.

so $p(O, n) = \#$ of partitions of n into odd parts.

$n = 6$ Partitions of n into odd parts:

$5+1, \quad 3+3, \quad 3+1+1+1, \quad 1+1+1+1+1+1$

so $p(O, 6) = 4$.

② $p(`\mathbb{Z}', n) = p(n)$.

Theorem ~~For $|q|<1$~~, Let $H \subseteq \mathbb{Z}^+$

$$\sum_{n=0}^{\infty} p(`H', n)\, q^n = \prod_{k \in H} \frac{1}{1-q^k} \qquad \text{for } |q|<1.$$

Defn Let $H \subset \mathbb{Z}^+$ & $d \geq 1$.

Let $p(\text{“}H\text{”}(\leq d), n) = $ # of partitions of n into parts which are elements of H each part appearing no more than d times.

Theorem

$$\sum_{n=0}^{\infty} p(\text{“}H\text{”}(\leq d), n)\, q^n = \prod_{m \in H} (1 + q^m + q^{2m} + \cdots + q^{dm})$$

$$= \prod_{m \in H} \frac{(1 - q^{(d+1)m})}{(1 - q^m)} \quad \text{for } |q| < 1.$$

~~Theorem~~

~~Lets~~

Cor. $$\sum_{n=0}^{\infty} \underbrace{p(\text{“}H\text{”}(\leq 1), n)}_{\text{\# of partitions of } n \text{ into distinct parts from } H.}\, q^n = \prod_{m \in H} (1 + q^m), \text{ for } |q| < 1.$$

Theorem (Euler)

Let $p(D, n) = $ # of partitions of n into distinct parts

Let $p(O, n) = $ # of partitions of n into odd parts.

Then

$$p(D, n) = p(O, n) \quad \text{for all } n.$$

(13)

Proof

$$\sum_{n=0}^{\infty} p(D,n) q^n = \prod_{n=1}^{\infty} (1+q^n) \quad \text{for } |q|<1.$$

$$\sum_{n=0}^{\infty} p(O,n) q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^{2n-1}} \quad \text{for } |q|<1.$$

$$\prod_{n=1}^{\infty} (1+q^n) = \prod_{n=1}^{\infty} \frac{(1+q^n)(1-q^n)}{(1-q^n)}$$

$$= \prod_{n=1}^{\infty} \frac{(1-q^{2n})}{(1-q^n)} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{2n-1})}.$$

Hence,

$$\sum_{n=0}^{\infty} p(D,n) q^n = \sum_{n=0}^{\infty} p(O,n) q^n \quad \text{for } |q|<1$$

and

$$p(D,n) = p(O,n) \quad \text{for all } n \geq 0.$$

Example: Verify $p(D,n) = p(O,n)$ for $n=9$.

Ptns of 9 into distinct parts	Ptns of 9 into odd parts
9	9
8+1	7+1+1
7+2	5+3+1
6+3	5+1+1+1+1
6+2+1	3+3+3
5+4	3+3+1+1+1
5+3+1	3+1+1+1+1+1+1
4+3+2	1+1+1+1+1+1+1+1+1
$p(D,9)=8$	$p(O,9)=8$

Theorem (Glaisher)(1883) Let $d \geq 1$.
Let N_d = set of positive integers not divisible by d.
Let N = set of positive integers.
Then

$$p(\text{`}N_{d+1}\text{'}, n) = p(\text{`}N\text{'}(\leq d), n) \quad \text{for all } n.$$

ie The # of partitions of n whose parts are not divisible by $(d+1)$
= # of partitions of n whose each part occurs $\leq d$ times.

Proof:
$$\sum_{n=0}^{\infty} p(\text{`}N\text{'}(\leq d), n) q^n = \prod_{n=1}^{\infty} \frac{(1-q^{(d+1)n})}{(1-q^n)}$$
$$= \prod_{\substack{n=1 \\ (d+1) \nmid n}}^{\infty} \frac{1}{1-q^n}$$
$$= \sum_{n=0}^{\infty} p(\text{`}N_{d+1}\text{'}, n) q^n, \quad \text{for } |q|<1,$$
& the result follows.

Graphical Representation of Partitions

If $n = \lambda_1 + \lambda_2 + \cdots + \lambda_r$, $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_r$ is a partition then we form the Ferrers graph of this partition by rows of $\lambda_1, \lambda_2, \ldots, \lambda_r$ dots:

$\bullet\ \bullet\ \bullet\ \bullet\ \bullet\ \bullet\ \bullet\ \bullet \leftarrow \lambda_1$ dots
$\bullet\ \bullet\ \bullet\ \bullet \leftarrow \lambda_2$ dots
$\vdots$
$\bullet\ \bullet\ \bullet \leftarrow \lambda_r$ dots

(15)

<u>Example</u> The Ferrers graph of $8+5+5+3+1$ is

$$\begin{array}{llllllll} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet \end{array}$$

<u>Defn</u>: Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ we form a new partition $\lambda' = (\lambda_1', \lambda_2', \ldots, \lambda_s')$ (where $s = \lambda_1$) called <u>conjugate</u> of λ by reading the columns of the Ferrer's graph of λ.

<u>Example</u> $\lambda = (8, 5, 5, 3, 1)$

$\lambda' = (5, 4, 4, 3, 3, 1, 1, 1)$.

<u>Theorem</u> Let λ be a partition. Then

(i) $|\lambda| = |\lambda'|$.

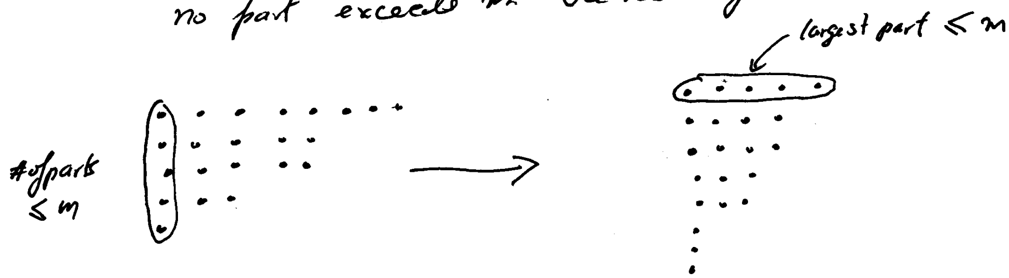
(ii) $(\lambda')' = \lambda$.

(iii) The map $\lambda \longmapsto \lambda'$ is a bijection of the set of ptns of n to itself.

<u>Theorem</u> Let $m, n \geq 1$.

The number of partitions of n with at most m parts $=$ the number of partitions of n in which no part exceeds m.

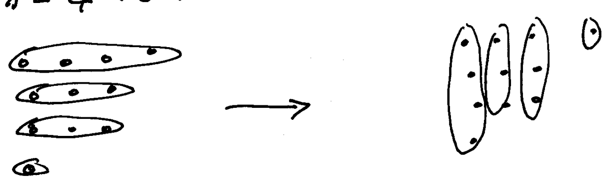
<u>Proof</u> Let $m, n \geq 1$.
The map $\lambda \mapsto \lambda'$ is a bijection from the set of partitions of n into at most m parts to the set of partitions in which no part exceeds m. The result follows.



$\square$

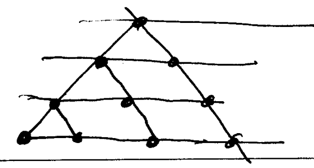
~~Theorem~~
<u>Definition</u> A partition λ is <u>self-conjugate</u> if $\lambda' = \lambda$

<u>Example</u>
$\lambda = 4 + 3 + 3 + 1$ is self-conjugate.



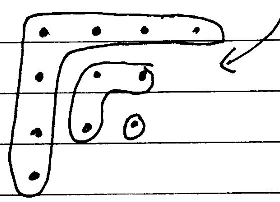
<u>Theorem</u> Let $n \geq 1$.
The number of self-conjugate partitions of n
= The number of partitions of n into distinct odd parts.

(17)



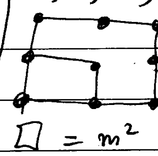
Example

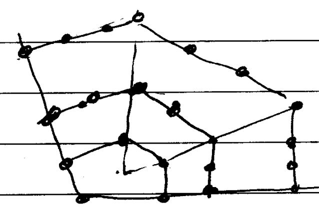
$(7, 3, 1)$



Δ numbers $= m(m+1)/2$

$1, 3, 6, 10$





$\square = m^2$

$1, 4, 9, 16, \ldots \quad 1 + 4 + 7 + \ldots + 3m-2$

$= \frac{3}{2}m^2 - \frac{m}{2}$ Pentag Numbers.

Euler (1750)
Legendre (1830)
Jacobi (1846)
Franklin (1881)

Theorem Let

$p_e(D, n) = \#$ of partitions of n into an even number of distinct parts

$p_o(D, n) = \#$ of partitions of n into an odd number of distinct parts

Then

$$p_e(D,n) - p_o(D,n) = \begin{cases} (-1)^m & \text{if } n = \frac{m(3m \pm 1)}{2} \\ 0 & \text{otherwise} \end{cases}$$

m	$\frac{m(3m-1)}{2}$	$\frac{m(3m+1)}{2}$
0	0	0
1	1	2
2	5	7
3	12	15
4	22	26

<u>Example</u>: $n=9$

Partitions into distinct parts:

9 $\qquad\qquad$ $p_e(D,9) = 4$

$8+1$

$7+2$ $\qquad\qquad$ $p_o(D,9) = 4$

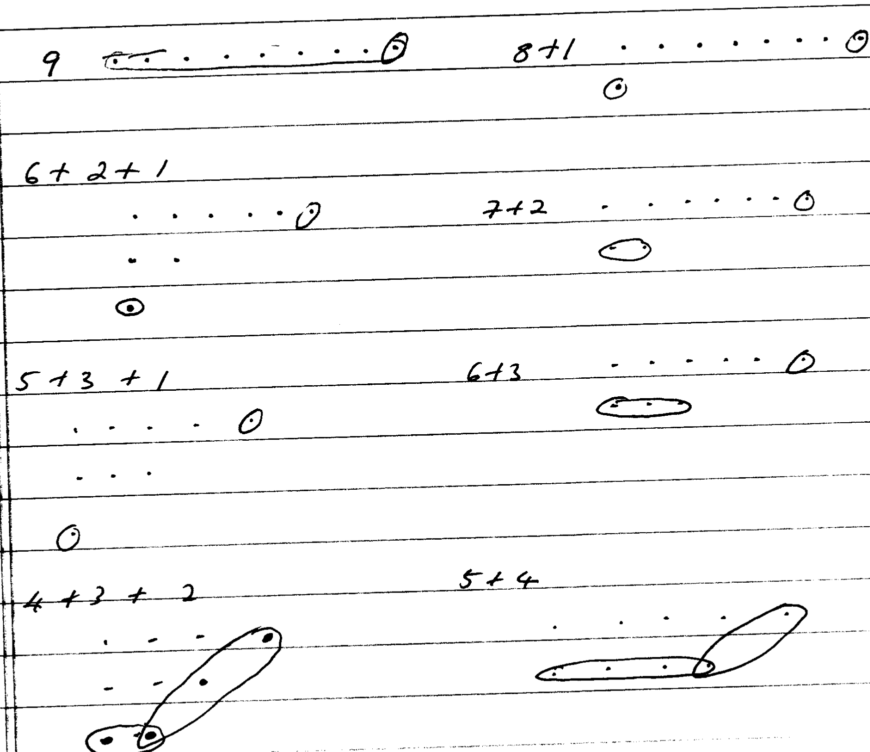
$6+3$

$6+2+1$

$5+4$

$5+3+1$

$4+3+2$



$n = 12$ (19)

Partitions of 12 into distinct parts:

12

11+1

10+2

9+3

9+2+1

8+4

8+3+1

7+5

7+4+1

7+3+2

6+5+1

6+4+2

6+3+2+1

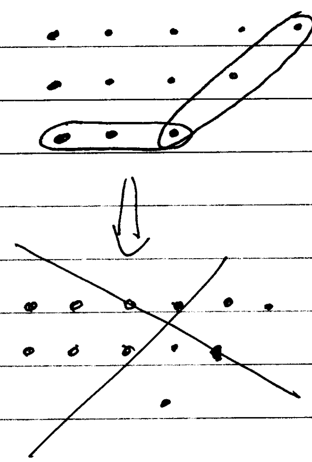
5+4+3

5+4+2+1

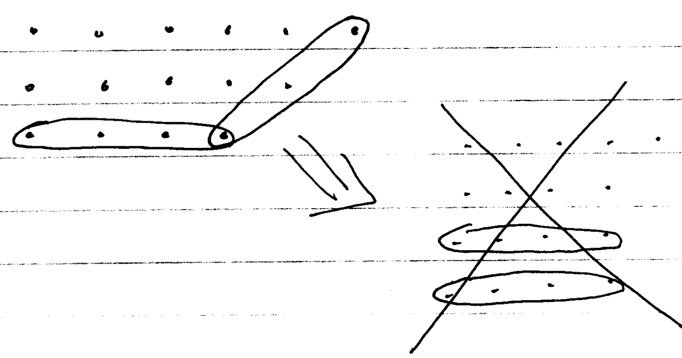
$p_o(D, 12) = 8$

$p_e(D, 12) = 7$

5+4+3



$\underline{n=15}$:

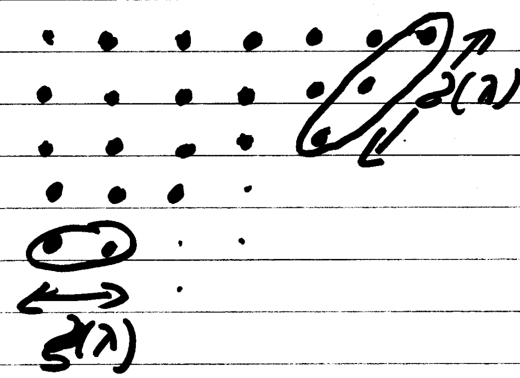


Proof of Theorem

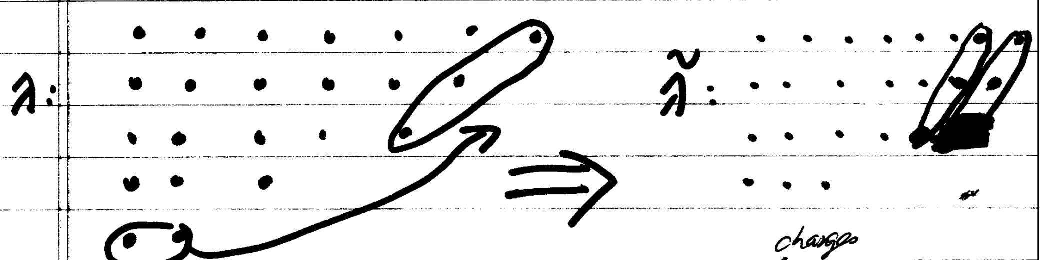
We define a map on the set of partitions of n into distinct parts. For a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ we let

$$s(\lambda) = \text{the smallest part } \lambda_r,$$

and $\quad \partial(\lambda) = \#$ of consecutive parts starting with the largest part.



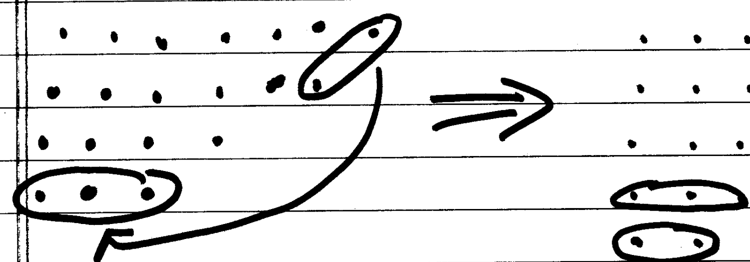
Case 1: $s(\lambda) \leq \partial(\lambda)$ $\qquad$ Note: $s(\tilde{\lambda}) > \partial(\tilde{\lambda})$



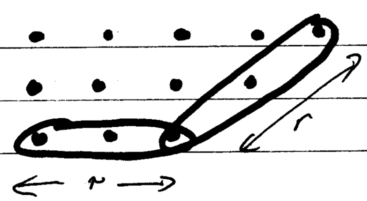
This changes the # of parts by 1, and hence changes the parity of the # of parts.

(21)

Case 2: $s(\lambda) > \sigma(\lambda)$ Note: $s(\tilde{\lambda}) \leq \sigma(\tilde{\lambda})$



Also changes the number of parts by 1, & hence changes the parity of the # of parts. Clearly this map is an involution, & hence one to one.
This map breaks down in to cases.
In Case 1 it breaks down when $r=\sigma(\lambda)=s(\lambda)$ and there is a node in common:



This is the partition (with r parts)

$$r + (r+1) + \dots + (r+(r-1))$$

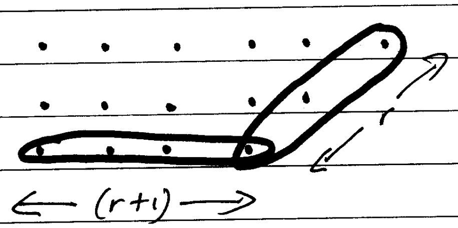
$$= r^2 + 1 + \dots (r-1) = r^2 + \tfrac{1}{2}r(r-1)$$

$$= \tfrac{3}{2}r^2 - \tfrac{1}{2}r = \tfrac{r}{2}(3r-1).$$

(22)

Case 2 breaks down when there is a node in common & $s(\pi) = r+1$ & $\partial(\pi) = r$.

This



The image $\tilde{\pi}$ will not be a ptn into distinct parts

This partition (with r parts) is

$$(r+1) + (r+2) + \cdots (r+r)$$
$$= r^2 + (1 + \cdots + r)$$
$$= r^2 + \tfrac{1}{2} r(r+1) = \frac{3r^2}{2} + \frac{r}{2} = \frac{r}{2}(3r+1).$$

Hence if n is not of the form $\frac{r}{2}(3r \pm 1)$ then there are no exceptions &

$$p_e(D, n) = p_o(D, n).$$

If $n = \frac{r}{2}(3r \pm 1)$ there is one exceptional partition. If r is even this exception has an even no of parts & $p_e(D, n) - p_o(D, n) = 1$.

If r is odd this one exception has odd # of parts &

$$p_e(D, n) - p_o(D, n) = -1.$$

So

$$p_e(D, n) - p_o(D, n) = (-1)^r \quad \text{if} \quad n = \tfrac{r}{2}(3r \pm 1).$$

$\square$

Euler's Pentagonal Number Theorem

For $|q|<1$,

$$\prod_{n=1}^{\infty}(1-q^n) = 1+\sum_{m=1}^{\infty}(-1)^m q^{\frac{1}{2}m(3m-1)}(1+q^m)$$
$$= \sum_{m=-\infty}^{\infty}(-1)^m q^{\frac{1}{2}m(3m-1)}.$$

Proof:

$$\sum_{m=-\infty}^{\infty}(-1)^m q^{\frac{1}{2}m(3m-1)}$$
$$= \sum_{m=0}^{\infty}(-1)^m q^{\frac{1}{2}m(3m-1)} + \sum_{m=-1}^{-\infty}(-1)^m q^{\frac{1}{2}m(3m-1)}$$
$$= 1+\sum_{m=1}^{\infty}(-1)^m q^{\frac{1}{2}m(3m-1)} + \sum_{k=1}^{\infty}(-1)^{-k} q^{\frac{1}{2}(-k)(-3k-1)}$$
$$= 1+\sum_{m=1}^{\infty}(-1)^m q^{\frac{1}{2}m(3m-1)} + \sum_{m=1}^{\infty}(-1)^m q^{\frac{1}{2}m(3m+1)}$$
$$= 1+\sum_{m=1}^{\infty}(-1)^m q^{\frac{1}{2}m(3m-1)}\left(1+q^m\right).$$
$$= 1+\sum_{n=1}^{\infty}\left(p_e(D,n)-p_o(D,n)\right)q^n$$

$$\prod_{n=1}^{m}(1-q^n) = (1+(-1)q)(1+(-1)q^2)\cdots(1+(-1)q^m)$$
$$= \left(\sum_{a_1=0}^{1}(-1)^{a_1}q^{a_1}\right)\left(\sum_{a_2=0}^{1}(-1)^{a_2}q^{2a_2}\right)\cdots\left(\sum_{a_m=0}^{1}(-1)^{a_m}q^{ma_m}\right)$$

(24)

$$= \sum_{a_1, a_2, \ldots, a_m \in \{0,1\}} (-1)^{a_1+a_2+\cdots+a_m} q^{a_1+2a_2+\cdots+ma_m}$$

Every partition of n into distinct parts $\leq m$ can be written unique as

$$n = a_1 + 2a_2 + \cdots + ma_m,$$

where $a_i \in \{0,1\}$. $a_i = 1$ iff i occurs as a part.
& $a_1 + a_2 + \cdots + a_m = \#$ of parts.
So let

$p_{e,m}(n) = \#$ of partitions of n into distinct even # of parts $\leq m$

$p_{o,m}(n) =$ ———
odd # ———.

Hence

$$\prod_{n=1}^{m} (1-q^n) = 1 + \sum_{n=1}^{\infty} (p_{e,m}(n) - p_{o,m}(n)) q^n$$

Letting $m \to \infty$ we find

$$\prod_{n=1}^{\infty} (1-q^n) = 1 + \sum_{n=1}^{\infty} (p_e(n) - p_o(n)) q^n. \quad \boxtimes$$

In fact,

$$\prod_{n=1}^{m} (1-q^n) = 1 + \sum_{n=1}^{m} (p_e(n) - p_o(n)) q^n + R_m(q)$$

where $$|R_m(q)| = \left| \sum_{n=m+1}^{\infty} p_{e,m}(n) - p_{o,m}(n) \, q^n \right|$$

$$\leq 2 \sum_{n=m+1}^{\infty} p(n) q^n \longrightarrow 0 \text{ as } m \to \infty. \; \square$$

(25)

Cor. (Euler)

Let $n \geq 1$.

$$p(n) - p(n-1) - p(n-2) + p(n-5) + p(n-7) + - \cdots + (-1)^m p(n - \tfrac{1}{2}m(3m-1)) + (-1)^m p(n - \tfrac{1}{2}m(3m+1)) + - \cdots = 0$$

where $p(j) = 0$ if $j < 0$.

Proof

$$\left(\prod_{n=1}^{\infty} \frac{1}{1-q^n}\right) \prod_{n=1}^{\infty} (1-q^n) = 1 \quad \text{for } |q|<1.$$

$$1 = (1 + p(1)q^1 + \cdots + p(n)q^n + \cdots)(1 - q - q^2 + q^5 + q^7 - + \cdots + (-1)^m q^{\frac{m}{2}(3m\mp 1)} + (-1)^m q^{\frac{m}{2}(3m+1)} + \cdots)$$

Coeff of q^n:

$$0 = p(n) - p(n-1) - p(n-2) + p(n-5) + p(n-7) - + \cdots + (-1)^m p(n - \tfrac{m}{2}(3m-1)) + (-1)^m p(n - \tfrac{m}{2}(3m+1)) + \cdots .$$

$\square$

Combinatorial Proof of (26)

$$p(\mathcal{D}, n) = p(\mathcal{O}, n) \quad \text{for all } n.$$

~~Suppose~~

~~$n = r_1 a_1 + r_2 a_2 + \cdots + r_s a_s$~~

~~is a partition~~

Suppose $\lambda = (\underbrace{a_1, a_1, \ldots, a_1}_{r_1}, \underbrace{a_2, \ldots, a_2}_{r_2}, \ldots\ldots, \underbrace{a_s, a_s, \ldots a_s}_{r_s})$ (1

is a partition of n into odd parts.

Let $r_i = \sum_j b_{ij} 2^j$

be the binary expansion of r_i where $b_{ij} = 0, 1$.

Then

$$n = \sum_{i=1}^{s} a_i r_i$$

$$= \sum_{i=1}^{s} a_i \sum_j b_{ij} 2^j$$

$$= \sum_{i=1}^{s} \sum_j b_{ij} 2^j a_i$$

This gives a partition of n into distinct parts with parts $2^j a_i$ (where $b_{ij} = 0, 1$).

Conversely, given given a ptn

$(c_1, c_2, \ldots, c_t)$ into distinct parts

write

each $c_k = 2^{e_k} d_k$ where d_k is odd.

Let $a_1, a_2, \dots, a_s$ be the different integers among (27)
$d_1, d_2, \dots, d_t$. For $1 \le i \le s$ let

$$b_{i,j} = \begin{cases} 1 & \text{if } 2^j a_i = \text{some } c_k \\ 0 & \text{otherwise.} \end{cases}$$

Then let

$$r_i = \sum_j b_{i,j}\, 2^j .$$

Then $n = \sum_{i=1}^{s} a_i r_i$ which reconstructs λ (*).
This defines a bijection between ptns of n into odd parts & distinct parts.

Example:

$\lambda = 9+9+5+5+5+3+3+1+1+1+1+1$

$$= \underbrace{(2)\,9}_{r_1} + \underbrace{(3)\,5}_{r_2} + \underbrace{2\,(3)}_{r_3} + \underbrace{5\,(1)}_{r_4}$$

$= (2)\,9 + (2+1)(5) + (2)(3) + (4+1)(1)$
$= 18 + 10 + 5 + 6 + 4 + 1$
$= 18 + 10 + 6 + 5 + 4 + 1$ ptn into distinct parts

Reverse:

$18 + 10 + 6 + 5 + 4 + 1$
$= 2(9) + 2(5) + 2(3) + 5 + 4(1) + 1$
$= 2(9) + (2+1)(5) + 2(3) + (4+1)\,1$

(28)

Example (Subbarao)

The number of partitions of n in which each part appears 2, 3 or 5 times equals the number of ptns of n into parts $\equiv$ 2, 3, 6, 9 or 10 mod 12.

Let $a(n)$ = # of ptns of n into in which each part appears 2, 3 or 5 times.

Let $b(n)$ = # of ptns of n into parts $\equiv$ 2, 3, 6, 9 or 10 mod 12.

For $|q|<1$,

$$\sum_{n=0}^{\infty} a(n) q^n = (1 + q^{1+1} + q^{1+1+1} + q^{1+1+1+1+1})$$

$$\cdot (1 + q^{2+2} + q^{2+2+2} + q^{2+2+2+2+2})$$

$$\cdot (1 + q^{3+3} + q^{3+3+3} + q^{3+3+3+3+3})$$

$$\vdots$$

$$= \prod_{n=1}^{\infty} (1 + q^{2n} + q^{3n} + q^{5n}).$$

$$= \prod_{n=1}^{\infty} ((1+q^{2n}) + q^{3n}(1+q^{2n}))$$

$$= \prod_{n=1}^{\infty} (1+q^{2n})(1+q^{3n}) = \prod_{n=1}^{\infty} \frac{(1+q^{2n})(1+q^{3n})(1-q^{2n})(1-q^{3n})}{(1-q^{2n})(1-q^{3n})}$$

$$= \prod_{n=1}^{\infty} \frac{(1-q^{4n})(1-q^{6n})}{(1-q^{2n})(1-q^{3n})}$$

$$= \frac{\displaystyle\prod_{\substack{n=1\\ n\equiv 0 \,(\mathrm{mod}\, 4)}}^{\infty} (1-q^n) \prod_{\substack{n=1\\ n\equiv 0 \,(\mathrm{mod}\, 6)}}^{\infty} (1-q^n)}{\displaystyle\prod_{\substack{n=1\\ n\equiv 0 \,(\mathrm{mod}\, 2)}}^{\infty} (1-q^{2n}) \prod_{\substack{n=1\\ n\equiv 0 \,(\mathrm{mod}\, 3)}}^{\infty} (1-q^n)}$$

$$= \frac{\displaystyle\prod_{\substack{n\geq 1\\ n\equiv \cancel{0},\cancel{4},\cancel{8} \,(\mathrm{mod}\, 12)}} (1-q^n) \prod_{\substack{n\geq 1\\ n\equiv \cancel{0},\cancel{6} \,(\mathrm{mod}\, 12)}} (1-q^n)}{\displaystyle\prod_{\substack{n\geq 1\\ n\equiv \cancel{0},2,\cancel{4},6,\cancel{8},10 \,(\mathrm{mod}\, 12)}} (1-q^n) \prod_{\substack{n\geq 1\\ n\equiv \cancel{0},3,\cancel{6},9 \,(\mathrm{mod}\, 12)}} (1-q^n)}$$

$$= \frac{1}{\displaystyle\prod_{\substack{n\geq 1\\ n\equiv 2,3,6,9,10 \,(\mathrm{mod}\, 12)}} (1-q^n)}$$

$$= \sum_{n=0}^{\infty} b(n) q^n.$$

Hence $a(n) = b(n)$ for $n \geq 0$.

(30)

An Upper bound for $p(n)$

$$p(n) \sim \frac{e^{K\sqrt{n}}}{4n\sqrt{3}} \quad \text{as } n \to \infty \quad (\text{Hardy, Ramanujan})$$

where $K = \pi\sqrt{\frac{2}{3}}$.

<u>Theorem</u>: If $n \geq 1$,

$$p(n) < e^{K\sqrt{n}} \quad \text{where } K = \pi\sqrt{\tfrac{2}{3}}.$$

<u>Proof</u>: Let $0 < q < 1$.

$$1 + \sum_{n=1}^{\infty} p(n) q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n} =: F(q)$$

$$q^n p(n) < F(q)$$

$$n \log q + \ln p(n) < \ln F(q) = -\sum_{n=1} \log(1 - q^n)$$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(q^n)^m}{m}$$

$$= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{q^{mn}}{m} = \sum_{m=1}^{\infty} \frac{1}{m} \sum_{n=1}^{\infty} q^{mn}$$

$$= \sum_{m=1}^{\infty} \frac{1}{m} \frac{q^m}{1-q^m}$$

$$m q^{m-1} < \frac{1-q^m}{1-q} = 1 + q + \cdots + q^{m-1} < m$$

(since $0 < q < 1$)

and $\quad m q^{m-1}(1-q) < 1 - q^m < m(1-q)$

$$\frac{m(1-q)}{q} < \frac{1-q^m}{q^m} < \frac{m(1-q)}{q^m}$$

(31)

$$\frac{q^m}{m(1-q)} < \frac{q^m}{1-q^m} < \frac{q}{m(1-q)},$$

&

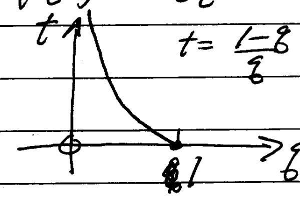
$$\frac{1}{m^2}\,\frac{q^m}{(1-q)} < \frac{1}{m}\,\frac{q^m}{1-q^m} < \frac{q}{m^2(1-q)}$$

so

$$\log F(q) = \sum_{m=1}^{\infty} \frac{1}{m}\,\frac{q^m}{1-q^m} < \frac{q}{1-q}\sum_{m=1}^{\infty}\frac{1}{m^2}$$

$$= \frac{q}{1-q}\,\zeta(2) = \left(\frac{q}{1-q}\right)\left(\frac{\pi^2}{6}\right) = \frac{\pi^2}{6t}$$

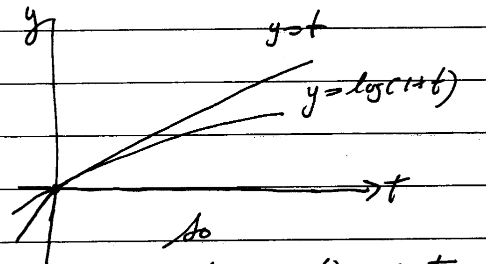
where $t = \frac{1-q}{q} = \frac{1}{q} - 1$



$t = \frac{1-q}{q}$

Now,

$$\ln p(n) < \log F(q) - n\log q = \log F(q) + n\ln\frac{1}{q}$$



$f(t) = -\log(1+t) + t$

$f'(t) = 1 - \frac{1}{1+t} = \frac{t}{1+t} > 0$

for $t > 0$.

So

$$\log(1+t) < t$$

$1 + t = \frac{1}{q}$, &

$$\ln\frac{1}{q} = \ln(1+t) < t$$

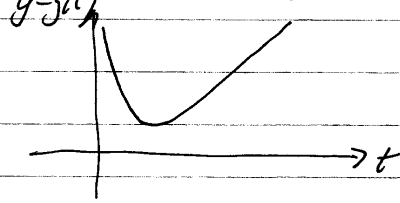
(32)

Hence,

$$\ln p(n) < \log F(q) + n \ln \frac{1}{q}$$

$$< \frac{\pi^2}{6t} + nt \quad \text{for all } t > 0.$$

Let $\quad g(t) = \frac{\pi^2}{6t} + nt \qquad t > 0.$



$$g'(t) = n \cancel{t} - \frac{\pi^2}{6t^2} = 0$$

when $\quad n - \frac{\pi^2}{6t^2} \quad$ ie

$$n = \frac{\pi^2}{6t^2} \quad \& \quad nt = \frac{\pi^2}{6t}$$

ie $\quad nt^2 = \frac{\pi^2}{6}$

$$t = \frac{1}{\sqrt{n}} \frac{\pi}{\sqrt{6}} \qquad \& \quad nt = \frac{\pi}{\sqrt{6}} \sqrt{n}$$

$\therefore \quad \ln p(n) < 2nt = \frac{2\pi\sqrt{n}}{\sqrt{6}} = \sqrt{\frac{2}{3}}\, \pi \sqrt{n}$

&

$$p(n) < e^{\pi\sqrt{\frac{2}{3}}\sqrt{n}}. \quad \square$$

(33)

Theorem

Suppose $\{a_n\}_{n=1}^{\infty} \subset \mathbb{Z}$. Then there exists a unique sequence $\{b_n\}_{n=1}^{\infty} \subset \mathbb{Z}$ such that formally

$$1 + \sum_{n=1}^{\infty} a_n q^n = \prod_{n=1}^{\infty} (1-q^n)^{b_n}$$

(*) ie $$1 + \sum_{n=1}^{m} a_n q^n = \prod_{n=1}^{m} (1-q^n)^{b_n} + O(q^{m+1})$$ for all $m \geq 1$.

Proof: We want

$$1 + a_1 q = (1-q)^{b_1} + O(q^2)$$

$$(1-q)^{b_1} = 1 - b_1 q + O(q^2)$$

so $b_1 = -a_1$.

Now assume (*) holds for $m = M$.

$$1 + \sum_{n=1}^{M} a_n q^n = \prod_{n=1}^{M} (1-q^n)^{b_n} + O(q^{M+1})$$

$$\cancel{= \prod_{n=1}^{M} (1-q^n)^{b_n} + c_{M+1} q^{M+1}}$$

$$\prod_{n=1}^{M} (1-q^n)^{b_n} = (1 + a_1 q + \cdots a_M q^M + \gamma q^{M+1} + \cdots)$$

(some $\gamma \in \mathbb{Z}$)

$$\prod_{n=1}^{M+1} (1-q^n)^{b_n} = (1 + a_1 q + \cdots + a_M q^M + \gamma q^{M+1} + \cdots)(1-q^{M+1})^{b_{M+1}}$$

$$= 1 + a_1 q + \cdots + a_M q^M + \gamma q^{M+1} + \cdots$$
$$- b_{M+1} q^{M+1} + \cdots.$$

We require

$$\gamma - b_{M+1} = a_{M+1}$$

& $b_{M+1} = \gamma - a_{M+1}$. Result by induction.

(34)

$$
\boxed{
\begin{array}{l}
\text{Let} \quad b_1 = -a_1. \\
\text{Given} \quad b_1, \ldots, b_{m-1} \\
\text{Let} \\
\quad g_m = \text{Coeff of } q^m \text{ in } \prod_{n=1}^{m-1} (1-q)^{b_n} \\
\text{Then} \\
\quad b_m = g_m - a_m.
\end{array}
}
$$

Any implementation is given as prodmake(QS, q, T) in the file FUNCS.txt.

(35)

```
Example (4)  Using MAPLE find formal products for GFs of
===========
           (i) p(n)
           (ii) p(D,n)
           (iii) The number of partitions of n in which difference
                 between parts is at least 2,

           at least up to q^30.

==========================================================================
> with(combinat):
> read "FUNCS.txt":

# (i)

> GFP:=add(P(n)*q^n,n=0..30);
```

$$GFP := 1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + 11q^6 + 15q^7 + 22q^8 + 30q^9 + 42q^{10} + 56q^{11}$$
$$+ 77q^{12} + 101q^{13} + 135q^{14} + 4565q^{29} + 176q^{15} + 231q^{16} + 297q^{17} + 385q^{18}$$
$$+ 3718q^{28} + 490q^{19} + 627q^{20} + 792q^{21} + 1002q^{22} + 1255q^{23} + 1575q^{24} + 1958q^{25}$$
$$+ 2436q^{26} + 5604q^{30} + 3010q^{27}$$

```
> pmake(GFP,q,30);
```

$$1/((1 - q)\ (1 - q^2)\ (1 - q^3)\ (1 - q^4)\ (1 - q^5)\ (1 - q^6)\ (1 - q^7)\ (1 - q^8)\ (1 - q^9)$$
$$(1 - q^{10})\ (1 - q^{11})\ (1 - q^{12})\ (1 - q^{13})\ (1 - q^{14})\ (1 - q^{15})\ (1 - q^{16})\ (1 - q^{17})$$
$$(1 - q^{18})\ (1 - q^{19})\ (1 - q^{20})\ (1 - q^{21})\ (1 - q^{22})\ (1 - q^{23})\ (1 - q^{24})\ (1 - q^{25})$$
$$(1 - q^{26})\ (1 - q^{27})\ (1 - q^{28})\ (1 - q^{29})\ (1 - q^{30}))$$

```
# (ii)

> GFPDP:=add(PDP(n)*q^n,n=0..30);
```

$$GFPDP := 1 + q + q^2 + 2q^3 + 2q^4 + 3q^5 + 4q^6 + 5q^7 + 6q^8 + 8q^9 + 10q^{10} + 12q^{11}$$
$$+ 15q^{12} + 18q^{13} + 22q^{14} + 256q^{29} + 27q^{15} + 32q^{16} + 38q^{17} + 46q^{18} + 222q^{28}$$
$$+ 54q^{19} + 64q^{20} + 76q^{21} + 89q^{22} + 104q^{23} + 122q^{24} + 142q^{25} + 165q^{26} + 296q^{30}$$
$$+ 192q^{27}$$

```
> pmake(GFPDP,q,30);
```

$$1/((1 - q)\ (1 - q^3)\ (1 - q^5)\ (1 - q^7)\ (1 - q^9)\ (1 - q^{11})\ (1 - q^{13})\ (1 - q^{15})\ (1 - q^{17})$$
$$(1 - q^{19})\ (1 - q^{21})\ (1 - q^{23})\ (1 - q^{25})\ (1 - q^{27})\ (1 - q^{29}))$$

```
# (iii)

> GFPRR:=add(PRR(n)*q^n,n=0..30);
```

$$GFPRR := 1 + q + q^2 + q^3 + 2q^4 + 2q^5 + 3q^6 + 3q^7 + 4q^8 + 5q^9 + 6q^{10} + 7q^{11} + 9q^{12}$$
$$+ 10q^{13} + 12q^{14} + 102q^{29} + 14q^{15} + 17q^{16} + 19q^{17} + 23q^{18} + 91q^{28} + 26q^{19}$$

(36)

```
      20        21        22        23        24        25        26         30        27
+ 31 q    + 35 q    + 41 q    + 46 q    + 54 q    + 61 q    + 70 q    + 117 q    + 79 q

> pmake(GFPRR,q,30);
                   4         6         9          11          14          16          19          21
1/((1 - q) (1 - q ) (1 - q ) (1 - q ) (1 - q  ) (1 - q  ) (1 - q  ) (1 - q  ) (1 - q  )

          24          26          29
    (1 - q  ) (1 - q  ) (1 - q  ))
```

==

Let $R(n)$ = # of partitions of n ~~into part~~ in which diff. between parts is at least 2.

Eg $n=9$ PTNS:

9
8 + 1
7 + 2
6 + 3
5 + 3 + 1

So $R(9) = 5$.

It seems that

$$\sum_{n=0}^{\infty} R(n) q^n = \frac{1}{(1-q)(1-q^4)(1-q^6)(1-q^9)\cdots}$$

$$= \prod_{\substack{n\geq 1 \\ n\equiv 1,4 \pmod 5}} \frac{1}{(1-q^n)} \quad \text{for } |q|<1.$$

We will prove this later. This identity is called the 1st Rogers-Ramanujan Identity.

(37)

$\boxed{\text{Recurrences for computing } \{a_n\} \,\&\, \{b_n\}}$

Suppose $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty} \subset \mathbb{Z}$ and

$$1 + \sum_{n=1}^{\infty} a_n q^n = \prod_{n=1}^{\infty} (1-q^n)^{b_n} \quad \text{for } |q|<1,$$

and converges absolutely.

Let $A(q) := 1 + \sum_{n=1}^{\infty} a_n q^n$ for $|q|<1$.

Then

$$\log(A(q)) = \log \prod_{n=1}^{\infty} (1-q^n)^{b_n} = \sum_{n=1}^{\infty} b_n \log(1-q^n)$$

$$\frac{d}{dq} \log(A(q)) = \frac{d}{dq}\left(\sum_{n=1}^{\infty} b_n \log(1-q^n)\right)$$

$$\frac{A'(q)}{A(q)} = \sum_{n=1}^{\infty} \frac{(-b_n) n q^{n-1}}{1-q^n}$$

$$q A'(q) = -\left(\sum_{n=1}^{\infty} b_n \frac{n q^n}{1-q^n}\right) A(q)$$

$$\frac{q^n}{1-q^n} = \sum_{m=1}^{\infty} q^{mn} \quad \text{and}$$

$$\sum_{n=1}^{\infty} b_n \frac{n q^n}{1-q^n} = \sum_{n=1}^{\infty} b_n \sum_{m=1}^{\infty} n q^{mn}$$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} n b_n q^{mn} = \sum_{N=1}^{\infty} \left(\sum_{mn=N} n b_n\right) q^N$$

$$= \sum_{N=1}^{\infty} D_N q^N \quad \text{where } D_N = \sum_{d \mid N} d\, b_d .$$

Hence, (38)

$$q A'(q) = \sum_{n=1}^{\infty} n a_n q^n = \left(-\sum_{j=1}^{\infty} D_j q^j\right)\left(1+\sum_{m=1}^{\infty} a_m q^m\right)$$

Coeff of q^n: $(j+m=n)$ $(a_0 := 1)$

$$n a_n = -\sum_{j=1}^{n} D_j a_{n-j} \quad \text{where } D_j = \sum_{d|j} d b_d$$

<u>Theorem</u> Suppose $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty} \subset \mathbb{Z}$ &

$$1 + \sum_{n=1}^{\infty} a_n q^n = \prod_{n=1}^{\infty} (1-q^n)^{b_n}$$

holds formally. Then

(i) $$n a_n = -\sum_{j=1}^{n} D_j a_{n-j} \quad \text{where } D_j = \sum_{d|j} d b_d$$

for $n \geq 1$.

(ii) $$b_n = a_n - \frac{1}{n}\left(\sum_{\substack{d|n \\ d<n}} d b_d + \sum_{j=1}^{n-1} D_j a_{n-j}\right)$$

for $n \geq 1$.

<u>Proof.</u> $$D_n = \sum_{d|n} d b_d = n b_n + \sum_{\substack{d|n \\ d<n}} d b_d.$$

Hence $$n a_n = -\left(D_n a_0 + \sum_{j=1}^{n-1} D_j a_{n-j}\right)$$

$$n a_n = -n b_n - \sum_{\substack{d|n \\ d<n}} d b_d - \sum_{j=1}^{n-1} D_j a_{n-j}$$

<u>Note</u>: This gives a recursion for computing $\{a_n\}$ given $\{b_n\}$ & for computing $\{b_n\}$ given $\{a_n\}$.

(39)

<u>Corollary</u> Let $n \geq 1$,

$$n\, p(n) = \sum_{j=1}^{n} \sigma(j)\, p(n-j)$$

where $\sigma(j) = \sum_{d \mid j} d$.

<u>Proof</u> Let $a_n = p(n)$ for $n \geq 1$. Then

$b_n = -1$ for $n \geq 1$.

$$1 + \sum_{n=1}^{\infty} p(n) q^n = \prod_{n=1}^{\infty} (1-q^n)^{-1}.$$

$$D_j = \sum_{d \mid j} d\, b_d = -\sum_{d \mid j} d = -\sigma(j),$$

and by (i),

$$n a_n = -\sum_{j=1}^{n} D_j\, a_{n-j} \quad \&$$

$$n\, p(n) = \sum_{j=1}^{n} \sigma(j)\, p(n-j). \quad \square$$