q-SERIES AND PARTITIONS

References

① George E. Andrews, The Theory of Partitions, Cambridge Univ. Press, 1998

② G. Gasper and M. Rahman, Basic Hypergeometric Series, Cambridge Univ. Press, 1990

③ G.E. Andrews, R. Askey & R. Roy, Special Functions, Cambridge Univ. Press, 1999.

ADDITIVE NUMBER THEORY

Basic Problem Let $A = \{a_1, a_2, \ldots\}$ be a set of positive integers. For which $n \geq 1$ can n be written as a sum of integers from A. Let $A(n)$ = number of such representations.

Goldbach's Conjecture (1742) Every even $n > 4$ is the sum of two odd primes.

Example $6 = 3+3$, $8 = 5+3$, $10 = 7+3 = 5+5$.

Sums of squares Let $k \geq 2$. Let $r_k(n)$ be number of solutions to

$$n = x_1^2 + x_2^2 + \cdots + x_k^2$$

where each $x_i \in \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$ (set of integers), and order matters. Jacobi (1829) found simple exact formulas for $r_k(n)$ where $k = 2, 4, 6$ or 8. Example:

$$r_2(n) = 4\,(d_1(n) - d_3(n))$$

where $d_j(n)$ = # of (positive) divisors of n congruent to j (mod 4).