Example $n=5$.

$$5 = 2^2+1^2 = (-2)^2+1^2 = 2^2+(-1)^2 = (-2)^2+(-1)^2$$
$$= 1^2+2^2 = 1^2+(-2)^2 = (-1)^2+2^2 = (-1)^2+(-2)^2$$

So $r_2(5) = 8$.

$d_1(5) = 2$ since $1, 5 \mid 5$.

$d_3(5) = 0$.

Waring's Problem Let $k \geq 1$ be given. Determine if there is a integer s (depending only on k) such that the equation

$$n = x_1^k + x_2^k + \cdots + x_s^k$$

has solutions for every $n \geq 1$.

Example

Every $n \geq 1$ be be represented as a sum of 4 squares.

. . . . 9 cubes.

. . . . 19 fourth powers.

Let $g(k)$ be the least value of s.

$g(2) = 4$ [Lagrange (1770)]

$g(3) = 9$ Wiefenich (1909)

$g(4) = 19$ Balasubramanian, Deshouillers, Dress (1986)

$g(5) = 37$ Chen (1964).

Unrestricted partitions Let $n \geq 1$. An (unrestricted) partition of n is a representation of n as a sum of positive integers

$$n = a_1 + a_2 + \cdots + a_k.$$

Order of parts is irrelevant and k is not restricted. Usually the parts are arranged in descending order.