<u>Theorem</u> Suppose $a_n > -1$ for all n (or more generally $\mathrm{Re}\, a_n > -1$ for all n). The product $\prod_{n=1}^{\infty}(1+a_n)$ converges iff $\sum_{n=1}^{\infty} \log(1+a_n)$ converges.

<u>Theorem</u> The product $\prod_{n=1}^{\infty}(1+a_n)$ converges absolutely iff $\sum_{n=1}^{\infty} a_n$ converges absolutely.

<u>Example</u> $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges absolutely. So

$\prod_{n=1}^{\infty}\left(1+\frac{1}{n^2}\right)$ converges absolutely.

In fact,

$$\prod_{n=1}^{\infty}\left(1+\frac{1}{n^2}\right) = \frac{e^{\pi}+e^{-\pi}}{2\pi}.$$

<u>Theorem</u> (Euler) For $|q|<1$,

$$\sum_{n=0}^{\infty} p(n)\, q^n = \cancel{\prod_{n=1}^{\infty} \frac{1}{1-q^n}} \cdot \prod_{k=1}^{\infty} \frac{1}{1-q^k}.$$

<u>~~Proof~~ Discussion</u>.

For $|q|<1$,
$$\frac{1}{1-q} = \sum_{n=0}^{\infty} q^n = 1+q+q^2+q^3+\cdots.$$

$$\frac{1}{1-q^2} = \sum_{n=0}^{\infty} q^{2n} = 1+q^2+q^4+q^6+\cdots.$$

$$\frac{1}{1-q^k} = \sum_{n=0}^{\infty} q^{kn} = 1+q^k+q^{2k}+q^{3k}+\cdots$$