Let $F_m(q) := \prod_{k=1}^{m} \frac{1}{1-q^k}$, $F(q) = \prod_{k=1}^{\infty} \frac{1}{1-q^k}$

The product $\prod_{k=1}^{\infty}(1-q^k)$ converges absolutely since $\sum_{k=1}^{\infty} q^k$ converges absolutely for $|q|<1$. Hence the product

$$\prod_{k=1}^{\infty} \frac{1}{1-q^k} = \frac{1}{\prod_{k=1}^{\infty}(1-q^k)}$$ converges.

Now suppose $0 \leq q < 1$.

$$F_{m+1}(q) = \frac{1}{1-q^{m+1}} F_m(q) \geq F_m(q),$$

and $\{F_m(q)\}$ is an increasing sequence which converges to $F(q)$.

Hence $$0 < F_m(q) \leq F(q).$$

$$F_m(q) = \sum_{n=0}^{\infty} p_m(n) q^n$$

$$= \sum_{n=0}^{m} p_m(n) q^n + \sum_{n=m+1}^{\infty} p_m(n) q^n$$

$$= \sum_{n=0}^{m} p(n) q^n + \sum_{n=m+1}^{\infty} p_m(n) q^n$$

$$\sum_{n=0}^{m} p(n) q^n \leq F_m(q) \leq F(q)$$

Therefore the series $\sum_{n=0}^{\infty} p(n) q^n$ converges, and

$$\sum_{n=0}^{\infty} p(n) q^n \leq F(q)$$

Also,

$$p_m(n) \leq p(n) \quad \text{for all } n$$