Defn:

Let $\mathcal{S} \subset \mathcal{P}$.

$$p(\mathcal{S}, n) := \sum_{\substack{\pi \in \mathcal{S} \\ |\pi| = n}} 1 .$$

so that
$$\sum_{n=0}^{\infty} p(\mathcal{S}, n)\, q^n = \sum_{\pi \in \mathcal{S}} q^{|\pi|} \qquad \text{for } |q|<1.$$

Let $H \subseteq \mathbb{Z}^+$ (set of positive integers).

Let
$$`H' := \{ \pi = (\lambda_1, \ldots, \lambda_k) \in \mathcal{P} : k \geq 1 \ \& \text{ each } \lambda_i \in H \}$$

Thus
$$p(`H', n) = \#\text{ of partitions of } n \text{ whose parts are elements of } H.$$

Example: ① Let $\mathcal{O}$ = set of odd positive integers $= \{1, 3, 5, 7, \ldots\}$.

Let $O = `\mathcal{O}'$.

so $p(O, n) = \#$ of partitions of n into odd parts.

$n = 6$ Partitions of n into odd parts:

$5+1, \quad 3+3, \quad 3+1+1+1, \quad 1+1+1+1+1+1$

so $p(O, 6) = 4$.

② $p(`\mathbb{Z}', n) = p(n)$.

Theorem Let $H \subseteq \mathbb{Z}^+$

$$\sum_{n=0}^{\infty} p(`H', n)\, q^n = \prod_{k \in H} \frac{1}{1-q^k} \qquad \text{for } |q|<1.$$