Euler's Pentagonal Number theorem

For $|q|<1$,

$$\prod_{n=1}^{\infty}(1-q^n) = 1+\sum_{m=1}^{\infty}(-1)^m q^{\frac{1}{2}m(3m-1)}(1+q^m)$$

$$= \sum_{m=-\infty}^{\infty}(-1)^m q^{\frac{1}{2}m(3m-1)}.$$

Proof: $$\sum_{m=-\infty}^{\infty}(-1)^m q^{\frac{1}{2}m(3m-1)}$$

$$= \sum_{m=0}^{\infty}(-1)^m q^{\frac{1}{2}m(3m-1)} + \sum_{m=-1}^{-\infty}(-1)^m q^{\frac{1}{2}m(3m-1)}$$

$$= 1+\sum_{m=1}^{\infty}(-1)^m q^{\frac{1}{2}m(3m-1)} + \sum_{k=1}^{\infty}(-1)^{-k} q^{\frac{1}{2}(-k)(-3k-1)}$$

$$= 1+\sum_{m=1}^{\infty}(-1)^m q^{\frac{1}{2}m(3m-1)} + \sum_{m=1}^{\infty}(-1)^m q^{\frac{1}{2}m(3m+1)}$$

$$= 1+\sum_{m=1}^{\infty}(-1)^m q^{\frac{1}{2}m(3m-1)}(1+q^m).$$

$$= 1 + \sum_{n=\cancel{0}1}^{\infty}\left(p_e(D,n) - p_o(D,n)\right)q^n$$

$$\prod_{n=1}^{m}(1-q^n) = (1+(-1)q)(1+(-1)q^2)\cdots(1+(-1)q^m)$$

$$= \left(\sum_{a_1=0}^{1}(-1)^{a_1}q^{a_1}\right)\left(\sum_{a_2=0}^{1}(-1)^{a_2}q^{2a_2}\right)\cdots\left(\sum_{a_m=0}^{1}(-1)^{a_m}q^{ma_m}\right)$$