$$= \sum_{a_1, a_2, \ldots, a_m \in \{0,1\}} (-1)^{a_1 + a_2 + \cdots + a_m} q^{a_1 + 2a_2 + \cdots + m a_m}$$

Every partition of n into ~~at most m~~ distinct parts $\leq m$ can be written unique as

$$n = a_1 + 2a_2 + \cdots + m a_m,$$

where $a_i \in \{0,1\}$. $a_i = 1$ iff i occurs as a part.
& $a_1 + a_2 + \cdots + a_m = \#$ of parts.
So let

$$p_{e,m}(n) = \# \text{ of partitions of } n \text{ into distinct even } \# \text{ of parts } \leq m$$

$$p_{o,m}(n) = \text{———— odd } \# \text{ ————}.$$

Hence

$$\prod_{n=1}^{m} (1 - q^n) = 1 + \sum_{n=1}^{\infty} (p_{e,m}(n) - p_{o,m}(n)) q^n$$

Letting $m \to \infty$ we find

$$\prod_{n=1}^{\infty} (1 - q^n) = 1 + \sum_{n=1}^{\infty} (p_e(n) - p_o(n)) q^n. \quad \boxed{S}$$

In fact,

$$\prod_{n=1}^{m} (1 - q^n) = 1 + \sum_{n=1}^{m} (p_e(n) - p_o(n)) q^n + R_m(q)$$

where

$$|R_m(q)| = \left| \sum_{n=m+1}^{\infty} p_{e,m}(n) - p_{o,m}(n) \, q^n \right|$$

$$\leq 2 \sum_{n=m+1}^{\infty} p(n) q^n \longrightarrow 0 \text{ as } m \to \infty. \; \square$$