An Upper bound for $p(n)$

$$p(n) \sim \frac{e^{K\sqrt{n}}}{4n\sqrt{3}} \quad \text{as } n \to \infty \quad (\text{Hardy, Ramanujan})$$

where $K = \pi\sqrt{\frac{2}{3}}$.

Theorem: If $n \geq 1$,

$$p(n) < e^{K\sqrt{n}} \quad \text{where } K = \pi\sqrt{\frac{2}{3}}.$$

Proof: Let $0 < q < 1$.

$$1 + \sum_{n=1}^{\infty} p(n) q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n} =: F(q)$$

$$q^n p(n) < F(q)$$

$$n \ln q + \ln p(n) < \ln F(q) = -\sum_{n=1}^{\infty} \log(1-q^n)$$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(q^n)^m}{m}$$

$$= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{q^{mn}}{m} = \sum_{m=1}^{\infty} \frac{1}{m} \sum_{n=1}^{\infty} q^{mn}$$

$$= \sum_{m=1}^{\infty} \frac{1}{m} \frac{q^m}{1-q^m}$$

$$m q^{m-1} < \frac{1-q^m}{1-q} = 1 + q + \dots + q^{m-1} < m$$

(since $0 < q < 1$)

and $m q^{m-1}(1-q) < 1 - q^m < m(1-q)$

$$\frac{m(1-q)}{q} < \frac{1-q^m}{q^m} < m\frac{(1-q)}{q^m}$$