Theorem

Suppose $\{a_n\}_{n=1}^{\infty} \subset \mathbb{Z}$. Then there exists a unique sequence $\{b_n\}_{n=1}^{\infty} \subset \mathbb{Z}$ such that formally

$$1 + \sum_{n=1}^{\infty} a_n q^n = \prod_{n=1}^{\infty} (1-q^n)^{b_n}$$

$$(*) \quad \text{ie} \quad 1 + \sum_{n=1}^{m} a_n q^n = \prod_{n=1}^{m} (1-q^n)^{b_n} + O(q^{m+1}) \quad \text{for all } m \geq 1.$$

Proof: We want

$$1 + a_1 q = (1-q)^{b_1} + O(q^2)$$

$$(1-q)^{b_1} = 1 - b_1 q + O(q^2)$$

so $\quad b_1 = -a_1.$

Now assume $(*)$ holds for $m = M$.

$$1 + \sum_{n=1}^{M} a_n q^n = \prod_{n=1}^{M} (1-q^n)^{b_n} + O(q^{M+1})$$

$$\cancel{= \prod_{n=1}^{M} (1-q^n)^{b_n} + c_{M+1} q^{M+1}}$$

$$\prod_{n=1}^{M} (1-q^n)^{b_n} = (1 + a_1 q + \cdots a_M q^M + \gamma q^{M+1} + \cdots)$$

(some $\gamma \in \mathbb{Z}$)

$$\prod_{n=1}^{M+1} (1-q^n)^{b_n} = (1 + a_1 q + \cdots + a_M q^M + \gamma q^{M+1} + \cdots)(1-q^{M+1})^{b_{M+1}}$$

$$= 1 + a_1 q + \cdots + a_M q^M + \gamma q^{M+1} + \cdots$$

$$- b_{M+1} q^{M+1} + \cdots$$

We require

$$\gamma - b_{M+1} = a_{M+1}$$

$$\& \quad b_{M+1} = \gamma - a_{M+1}. \quad \text{Result by induction.}$$