Hence, (38)

$$qA'(q) = \sum_{n=1}^{\infty} n\, a_n q^n = \left(-\sum_{j=1}^{\infty} D_j q^j\right)\left(1+\sum_{m=1}^{\infty} a_m q^m\right)$$

Coeff of q^n: $(j+m=n)$ $\qquad (a_0 := 1)$

$$n\, a_n = -\sum_{j=1}^{n} D_j\, a_{n-j} \quad \text{where } D_j = \sum_{d|j} d\, b_d$$

<u>Theorem</u> Suppose $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty} \subset \mathbb{Z}$ &

$$1 + \sum_{n=1}^{\infty} a_n q^n = \prod_{n=1}^{\infty} (1-q^n)^{b_n}$$

holds formally. Then

(i) $$n\, a_n = -\sum_{j=1}^{n} D_j\, a_{n-j} \quad \text{where } D_j = \sum_{d|j} d\, b_d$$

for $n \geq 1$.

(ii) $$b_n = a_n - \frac{1}{n}\left(\sum_{\substack{d|n \\ d<n}} d\, b_d + \sum_{j=1}^{n-1} D_j\, a_{n-j}\right)$$

for $n \geq 1$.

<u>Proof.</u> $$D_n = \sum_{d|n} d\, b_d = n\, b_n + \sum_{\substack{d|n \\ d<n}} d\, b_d.$$

Hence $$n a_n = -\left(D_n a_0 + \sum_{j=1}^{n-1} D_j\, a_{n-j}\right)$$

$$n a_n = -n b_n - \sum_{\substack{d|n \\ d<n}} d b_d - \sum_{j=1}^{n-1} D_j\, a_{n-j}$$

<u>Note</u>: This gives a recursion for computing $\{a_n\}$ given $\{b_n\}$ & for computing $\{b_n\}$ given $\{a_n\}$.