Chapter 6 Sums of Squares

When $n \geq 0$, $(a)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$

$$= \frac{(a)_\infty}{(aq^n)_\infty}.$$

When $n = -m < 0$ we define

$$(a)_n := \frac{(a)_\infty}{(aq^n)_\infty} = \frac{(a)_\infty}{(aq^{-m})_\infty} = \frac{1}{(1-aq^{-m})\cdots(1-aq^{-1})}$$

$$= \frac{(-q/a)^m \, q^{m(m-1)/2}}{(q/a)_m}.$$

Bilateral Basic Series:

$${}_r\psi_r\left[\begin{matrix} a_1, a_2, \ldots a_r, \\ b_1, b_2, \ldots, b_r \end{matrix}; q, z\right] := \sum_{n=-\infty}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_r)_n}{(b_1)_n (b_2)_n \cdots (b_r)_n} z^n$$

$$= \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_r)_n \, z^n}{(b_1)_n (b_2)_n \cdots (b_r)_n}$$

$$+ \sum_{n=1}^{\infty} \frac{(q/b_1)_n (q/b_2)_n \cdots (q/b_r)_n}{(q/a_1)_n (q/a_2)_n \cdots (q/a_r)_n} \left(\frac{b_1 b_2 \cdots b_r}{a_1 a_2 \cdots a_r z}\right)^n$$

converges for $|q| < 1$ and

$$\left|\frac{b_1 b_2 \cdots b_r}{a_1 a_2 \cdots a_r}\right| < |z| < 1$$

Ramanujan's ${}_1\psi_1$ summation For $|q|<1$,

$$(*)\quad {}_1\psi_1\left(\begin{matrix}a\\b\end{matrix};q,z\right) = \sum_{n=-\infty}^{\infty} \frac{(a)_n}{(b)_n} z^n = \frac{(\frac{b}{a})_\infty (az)_\infty (\frac{q}{az})_\infty (q)_\infty}{(b)_\infty (q/a)_\infty (\frac{b}{az})_\infty (z)_\infty}$$

for $\left|\frac{b}{a}\right| < |z| < 1$.

We need

Lemma If $f(z)$ is analytic for $|z|<1$
and $f(a_n)=0$ for $n \geq 1$
~~where $\lim_{n\to\infty} a_n$~~ for infinitely many a_n where $a_n \to 0$ as $n\to\infty$,
then $f(z)=0$ for all $|z|<1$.

We need q-bin Thm:

$$\sum_{n=0}^{\infty} \frac{(a)_n}{(q)_n} t^n = \frac{(at)_\infty}{(t)_\infty} \qquad \text{for } |q|<1, |t|<1.$$

Proof of $(*)$ We show that $(*)$ holds for $b=q^N$
where N is a positive integer.

$${}_1\psi_1\left(\begin{matrix}a\\b\end{matrix};q,z\right) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} z^n + \sum_{n=1}^{\infty} \frac{(q/b)_n}{(q/a)_n} \left(\frac{b}{az}\right)^n$$

$${}_1\psi_1\left(\begin{matrix}a\\q^N\end{matrix};q,z\right) = \sum_{n=0}^{\infty} \frac{(a)_n}{(q^N)_n} z^n + \sum_{n=1}^{\infty} \frac{(q^{1-N})_n}{(q/a)_n} \left(\frac{b}{az}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{(a)_n z^n}{(q^N)_n} + \sum_{n=1}^{N-1} \frac{(q^{1-N})_n}{(q/a)_n} \left(\frac{b}{az}\right)^n$$

(since $(q^{1-N})_n = 0$ if $n \geq N$)

$$= \sum_{n=1-N}^{\infty} \frac{(a)_n z^n}{(q^N)_n}$$

$$= \sum_{n=0}^{\infty} \frac{(a)_{n+1-N}}{(q^N)_{n+1-N}} z^{n+1-N}$$

Now $(a)_{n+1-N} = \dfrac{(a)_\infty}{(aq^{n+1-N})_\infty} = \dfrac{(a)_\infty}{(aq^{1-N})_\infty} \dfrac{(aq^{1-N})_\infty}{(aq^{1-N+n})_\infty}$

$$= (a)_{1-N} \, (aq^{1-N})_n ,$$

and

$$(q^N)_{n+1-N} = (q^N)_{1-N} \, (q)_n .$$

Hence

$${}_1\psi_1\left(\begin{matrix} a \\ q^N \end{matrix}; q, z\right) = \frac{(a)_{1-N} z^{1-N}}{(q^N)_{1-N}} \sum_{n=0}^{\infty} \frac{(aq^{1-N})_n z^n}{(q)_n}$$

$$= \frac{(a)_{1-N} z^{1-N}}{(q^N)_{1-N}} \cdot \frac{(azq^{1-N})_\infty}{(z)_\infty} \qquad \text{(by } q\text{-bin Thm.)}$$

$$(azq^{1-N})_\infty = (1-azq^{1-N})\cdots(1-azq^{-1})(az)_\infty$$

$$= (-azq^{1-N})(1-a^{-1}z^{-1}q^{N-1})\cdots(-azq^{-1})(1-a^{-1}z^{-1}q)(az)_\infty$$

$$= (-az)^{N-1} q^{-N(N-1)/2} (a^{-1}z^{-1}q)_{N-1} (az)_\infty$$

$$= (-az)^{N-1} q^{-N(N-1)/2} \frac{(a^{-1}z^{-1}q)_\infty (az)_\infty}{(a^{-1}z^{-1}q^N)_\infty}$$

$$(a)_{1-N} = \frac{(a)_\infty}{(aq^{1-N})_\infty} = \frac{(a)_\infty (a^{-1}q^N)_\infty (-a)^{N-1} q^{N(N-1)/2}}{(a^{-1}q)_\infty (a)_\infty}$$

$$= \frac{(a^{-1}q^N)_\infty (-a)^{N-1} q^{N(N-1)/2}}{(a^{-1}q)_\infty}$$

$$(q^N)_{1-N} = \frac{(q^N)_\infty}{(q)_\infty}.$$

Hence,

$${}_1\psi_1\left(\begin{matrix} a \\ q^N \end{matrix}; q, z\right) = \frac{(a^{-1}q^N)_\infty}{(a^{-1}q)_\infty} \frac{(q)_\infty}{(q^N)_\infty} \frac{(a^{-1}z^{-1}q)_\infty (az)_\infty}{(a^{-1}z^{-1}q^N)_\infty (z)_\infty}$$

and (*) holds for $b = q^N$.
The result follows by the Lemma since both sides of (*) define an analytic function of b for $|b| < |az|$ that agrees for $b = q^N \to 0$ as $N \to \infty$. $\square$

Definition Let $k \geq 1$, $n \geq 0$.

Let $r_k(n)$ = number of ways of writing n as a sum of k squares

ie

$$r_k(n) = \{(x_1, x_2, \ldots, x_k) \in \mathbb{Z}^k : x_1^2 + x_2^2 + \cdots + x_k^2 = n\}$$

For example,

$r_2(1) = 4$ since

$1 = 0^2 + 1^2 = 1^2 + 0^2 = 0^2 + (-1)^2 = (-1)^2 + 0^2$.

Theorem (Jacobi) For $n \geq 1$,

$$r_2(n) = 4\left(\sum_{\substack{d \mid n \\ d \equiv 1 \pmod 4}} 1 - \sum_{\substack{d \mid n \\ d \equiv 3 \pmod 4}} 1\right).$$

Example: $r_2(1) = 4 = 4(1-0)$.

Lemma

$$\sum_{n=0}^{\infty} r_k(n) q^n = \left(\sum_{n=-\infty}^{\infty} q^{n^2}\right)^k$$

Proof: Let $|q|<1$.

$$\left(\sum_{n=-\infty}^{\infty} q^{n^2}\right)^k = \sum_{n_1 \in \mathbb{Z}} q^{n_1^2} \sum_{n_2 \in \mathbb{Z}} q^{n_2^2} \cdots \sum_{n_k \in \mathbb{Z}} q^{n_k^2}$$

$$= \sum_{(n_1, n_2, \ldots, n_k) \in \mathbb{Z}^k} q^{n_1^2 + \cdots + n_k^2}$$

$$= \sum_{n=0}^{\infty} \Big(\sum_{\substack{(n_1, n_2, \dots n_k) \in \mathbb{Z}^k \\ n_1^2 + \dots + n_k^2 = n}} 1 \Big) q^n$$

$$= \sum_{n=0}^{\infty} r_k(n)\, q^n. \quad \square$$

<u>Lemma:</u> $$\sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = \prod_{n=1}^{\infty} \frac{(1-q^n)}{(1+q^n)} = \frac{(q;q)_\infty}{(-q;q)_\infty}$$

for $|q|<1$.

This was proved in Ch2 using JTP (see p.21 of Notes).

<u>Proof of Theorem:</u>

$${}_1\psi_1\left(\begin{matrix}-1\\-q\end{matrix}; q, i\sqrt{q}\right) = \sum_{n=-\infty}^{\infty} \frac{(i\sqrt{q})^n (-1;q)_n}{(-q;q)_n}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(-1;q)_n}{(-q;q)_n} i^n q^{n/2}$$

$$+ \sum_{n=1}^{\infty} \frac{(-1;q)_n}{(-q;q)_n} \left(\frac{-q}{-1/(iq^{1/2})}\right)^n$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(-1;q)_n}{(-q;q)_n} (i^n + i^{-n}) q^{n/2}$$

$n \pmod 4$	$i^n + i^{-n}$	
0	2	
1	$i + i^{-1} = 0$	
2	-2	
3	$i^3 + i^{-3} = 0$	

$i^2 = -1$

$i = -i^{-1}$

So $i^{2m} + i^{-2m} = (-1)^m 2$

So ${}_1\psi_1\left(\begin{matrix}-1\\-q\end{matrix}; q, i\sqrt{q}\right) =$

$$= 1 + \sum_{n=1}^{\infty} \frac{(1+1)\cdots(1+q^{n-1})}{(1+q)\cdots(1+q^n)} (i^n + i^{-n}) q^{n/2}$$

$$= 1 + 2 \sum_{n=1}^{\infty} \frac{(i^n + i^{-n}) q^{n/2}}{1+q^n}$$

$$= 1 + 4 \sum_{n=1}^{\infty} \frac{q^n}{1+q^{2n}}$$

By Ramanujan's summation,

$$1 + 4 \sum_{n=1}^{\infty} \frac{(-1)^n q^n}{1+q^{2n}} = \frac{(q)_\infty (-i\sqrt{q})_\infty \left(\frac{q}{-i\sqrt{q}}\right)_\infty (q)_\infty}{(-q)_\infty (-q)_\infty \left(\frac{-q}{-i\sqrt{q}}\right)_\infty (i\sqrt{q})_\infty}$$

$$= \frac{(q)_\infty^2 (-i\sqrt{q})_\infty (i\sqrt{q})_\infty}{(-q)_\infty^2 (i\sqrt{q})_\infty (-i\sqrt{q})_\infty} = \left(\frac{(q)_\infty}{(-q)_\infty}\right)^2$$

~~Replacing q by~~

Hence
$$\left(\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2}\right)^2 = 1+4\sum_{n=1}^{\infty}\frac{(-1)^n q^n}{1+q^{2n}}.$$

Replacing q by $-q$ we have

$$\sum_{n=0}^{\infty} r_2(n)q^n = \left(\sum_{n=-\infty}^{\infty} q^{n^2}\right)^2 = 1+4\sum_{n=1}^{\infty}\frac{q^n}{1+q^{2n}}$$

$$= 1+4\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}(-1)^m q^{n+2mn}$$

$$= 1+4\sum_{m=0}^{\infty}(-1)^m\sum_{k=1}^{\infty} q^{k(2m+1)}$$

$$= 1+4\sum_{m=0}^{\infty}\left(\sum_{k=1}^{\infty} q^{k(4m+1)} - \sum_{k=1} q^{k(4m+3)}\right)$$

$$= 1+4\left(\sum_{n=1}^{\infty}\Big(\sum_{\substack{k(4m+1)=n\\ k\geq 1,\, 4m+1\geq 1}} 1\Big) q^n - \sum_{n=1}\Big(\sum_{\substack{k(4m+3)=n\\ k\geq 1,\, m\geq 0}} 1\Big) q^n\right)$$

$$= 1+4\left(\sum_{n=1}^{\infty}\Big(\sum_{\substack{d\mid n\\ d\equiv 1 \pmod 4}} 1 - \sum_{\substack{d\mid n\\ d\equiv 3 \pmod 4}} 1\Big) q^n\right.$$

Hence for $n\geq 1$,

$$r_2(n) = 4\left(\sum_{\substack{d\mid n\\ d\equiv 1 \pmod 4}} 1 - \sum_{\substack{d\mid n\\ d\equiv 3 \pmod 4}} 1\right) \qquad \square$$

Bailey's ${}_6\psi_6$ Summation (1936)

$${}_6\psi_6\left[\begin{matrix} q\sqrt{a}, -q\sqrt{a}, b, c, d, e \\ \sqrt{a}, -\sqrt{a}, \frac{aq}{b}, \frac{aq}{c}, \frac{aq}{d}, \frac{aq}{e} \end{matrix}; q, \frac{qa^2}{bcde}\right]$$

$$= \frac{(aq)_\infty \left(\frac{aq}{bc}\right)_\infty \left(\frac{aq}{bd}\right)_\infty \left(\frac{aq}{be}\right)_\infty \left(\frac{aq}{cd}\right)_\infty \left(\frac{aq}{ce}\right)_\infty \left(\frac{aq}{de}\right)_\infty (q)_\infty \left(\frac{q}{a}\right)_\infty}{\left(\frac{aq}{b}\right)_\infty \left(\frac{aq}{c}\right)_\infty \left(\frac{aq}{d}\right)_\infty \left(\frac{aq}{e}\right)_\infty \left(\frac{q}{b}\right)_\infty \left(\frac{q}{c}\right)_\infty \left(\frac{q}{d}\right)_\infty \left(\frac{q}{e}\right)_\infty \left(\frac{qa^2}{bcde}\right)_\infty}$$

if $\left|\frac{qa^2}{bcde}\right| < 1$, $(\& \, |q|<1)$.

Note: $\frac{(q\sqrt{a})_n}{(\sqrt{a})_n} \frac{(-q\sqrt{a})_n}{(-\sqrt{a})_n} = \frac{(1-\sqrt{a}q^n)(1+\sqrt{a}q^n)}{(1-\sqrt{a})(1+\sqrt{a})}$

$= \frac{(1-aq^{2n})}{(1-a)}$ for $n \geq 0$.

$${}_6\psi_6[\;] = \sum_{n=0}^{\infty} \left(\frac{1-aq^{2n}}{1-a}\right) \frac{(b)_n (c)_n (d)_n (e)_n}{\left(\frac{aq}{b}\right)_n \left(\frac{aq}{c}\right)_n \left(\frac{aq}{d}\right)_n \left(\frac{aq}{e}\right)_n} \left(\frac{qa^2}{bcde}\right)^n$$

$$+ \sum_{n=1}^{\infty} \left(\frac{1-\frac{1}{a}q^{2n}}{1-1/a}\right) \frac{(b/a)_n \left(\frac{c}{a}\right)_n \left(\frac{d}{a}\right)_n \left(\frac{e}{a}\right)_n}{\left(\frac{q}{b}\right)_n \left(\frac{q}{c}\right)_n \left(\frac{q}{d}\right)_n \left(\frac{q}{e}\right)_n} \left(\frac{qa^2}{bcde}\right)^n$$

$$\frac{-a^5 q^4}{bcde} \; \frac{1}{-q^2 abcde \cdot \frac{qa^2}{bcde}} = \frac{q a^2}{bcde}$$

<u>Application (1)</u> Let $b = a$, $d = c = e = -1$.

$$ {}_6\psi_6 = 1 + \sum_{n=1}^{\infty} \left(\frac{1-aq^{2n}}{1-a}\right) \frac{(a)_n}{(q)_n} \frac{(-1)_n}{(-aq)_n} \frac{(-1)_n}{(-aq)_n} \frac{(-1)_n}{(-aq)_n} \left(\frac{-qa}{c^2}\right)^n $$

$$ + \sum_{n=1}^{\infty} \frac{\left(1-\frac{1}{a}q^{2n}\right)}{(1-1/a)} \frac{(1)_n}{(q/a)_n} \frac{(-1/a)_n}{(q/c)_n} \frac{(-1/a)_n}{(-q/c)_n} \frac{(-1/a)_n}{(-q)_n} \left(\frac{-qa}{c^2}\right)^n $$

$$ \frac{(a)_n}{(1-a)(q)_n} = \frac{(1-a)\cdots(1-aq^{n-1})}{(1-a)\cdots(1-q^n)} = \frac{1}{1-q^n} \quad \text{for } n \geq 1 $$

Hence

$$ 1 + \sum_{n=1}^{\infty} \frac{(1-aq^{2n})}{(1-q^n)} \frac{(-1)_n^3}{(-aq)_n^3} (-aq)^n $$

$$ = \cancel{\frac{(aq)_\infty^2 (q)_\infty (q/a)_\infty}{(-aq)_\infty^4 (-q)_\infty^4 (qa^2)_\infty}} \; \frac{(aq)_\infty (-q)_\infty^3 (aq)_\infty^3 (q)_\infty (\frac{q}{a})_\infty}{(q)_\infty (-aq)_\infty^3 (\frac{q}{a})_\infty (-q)_\infty^3 (-\frac{q}{a})_\infty} $$

Let $a \to 1$ we obtain

$$ 1 + \sum_{n=1}^{\infty} (1+q^n) \frac{(-1)_n^3}{(-q)_n^3} (-q^n) = \left(\frac{(q)_\infty}{(-q)_\infty}\right)^4 $$

$$ 1 + 8 \sum_{n=1}^{\infty} \frac{(-1)^n q^n}{(1+q^n)^2} = \left(\frac{(q)_\infty}{(-q)_\infty}\right)^4 $$

(11)

$$(1-z)^{-1} = \frac{1}{1-z} = \sum_{m=0}^{\infty} z^m \quad (\text{for } |z|<1)$$

$$\frac{z}{1-z} = \sum_{m=1}^{\infty} m z^m$$

$$\frac{z}{1+z} = \sum_{m=1}^{\infty} m (-1)^{m+1} z^m$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{(-1)^n q^n}{(1+q^n)^2} = \sum_{n=1}^{\infty} (-1)^n \sum_{m=1}^{\infty} m (-1)^{m+1} q^{nm}$$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} m (-1)^{n+m+1} q^{nm}$$

It follows that

$$r_4(N) = 8 \sum_{\substack{n, m \geq 1 \\ nm = N}} m (-1)^{n+m+1+nm}$$

$$= 8 \sum_{d|N} d (-1)^{1+N+d+\frac{N}{d}}$$

Case 1 N is odd. Then if $d|N$, d is odd

Then $(-1)^{1+N+d+N/d} = 1$

and

$$r_4(N) = 8\sum_{d|N} d$$

Case 2 N is even, $N = m2^\alpha$ (m odd, $\alpha \geq 1$).

Any divisor (+ve) d' of N can be written uniquely as $d' = d2^j$ where $d|m$ & $0 \leq j \leq \alpha$.

Hence

$$r_4(N) = 8\sum_{d'|N} d'(-1)^{1+N+d'+N/d'}$$

$$= 8\sum_{d|m}\sum_{j=0}^{\alpha} d2^j(-1)^{1+N+d2^j+\left(\frac{m}{d}\right)2^{\alpha-j}}$$

$$= 8\sum_{d|m} d\left(1 + \sum_{j=1}^{\alpha-1} 2^j(-1) + 2^\alpha\right)$$

$$= 8\sum_{d|m} d(1+2) = 8\sum_{\substack{d|N \\ 4\nmid d}} d$$

since $1+2+\cdots+2^{\alpha-1} = 2^\alpha - 1$

& $2+\cdots+2^{\alpha-1} = (2^\alpha - 2)$.

Hence we have

Theorem (Jacobi) For $n \geq 1$,

$$r_4(n) = 8\sum_{\substack{d|n \\ 4\nmid d}} d$$

Corollary (Lagrange)
Every integer $n \geq 1$ can be written as the sum of four squares.

Application (2)
Let $b, c, d, e = -1$ and let $a \to 1$ we find (eventually) that

Theorem (Jacobi) For $n \geq 1$,

$$r_8(n) = 16(-1)^n \sum_{d|n} (-1)^d d^3.$$

Application (3)
Let $q \to q^5$, $a = q^4$, $b = c = q$ $d = e = q^3$.
We find (eventually) that

$$\sum_{n=0}^{\infty} \frac{q^{5n+1}}{(1-q^{5n+1})^2} - \frac{q^{5n+2}}{(1-q^{5n+2})^2} - \frac{q^{5n+3}}{(1-q^{5n+3})^2} + \frac{q^{5n+4}}{(1-q^{5n+4})^2}$$

$$= q \frac{(q^5; q^5)_\infty^5}{(q)_\infty} \quad \text{(Ramanujan)}$$

Hence we have

$$\sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\frac{q^n}{(1-q^n)^2} = \sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\sum_{m=1}^{\infty} m q^{mn}$$

$$= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{n}{5}\right) m q^{mn}$$

$$= \sum_{N=1}^{\infty}\left(\sum_{\substack{nm=N\\ n,m\geq 1}}\left(\frac{n}{5}\right) m\right) q^N$$

$$= \sum_{N=1}^{\infty}\left(\sum_{d\mid N}\left(\frac{d}{5}\right)\frac{N}{d}\right) q^N$$

Here, $\left(\frac{n}{5}\right) = \begin{cases} 0 & \text{if } n\equiv 0 \pmod 5 \\ 1 & \text{if } n \equiv 1,4 \pmod 5 \\ -1 & \text{if } n\equiv 2,3 \pmod 5\end{cases}$

is the Legendre symbol mod 5.

Let $a_t(n)$ = # of partitions of n that are t-cores (i.e. have no hooknumbers that are multiples of t).

Then recall that

$$\sum_{n=0}^{\infty} a_t(n) q^n = \frac{(q^t;q^t)_\infty^t}{(q)_\infty} \quad \text{for } |q|<1.$$

Hence, $$\sum_{n=0}^{\infty} a_5(n) q^{n+1} = q\frac{(q^5;q^5)_\infty^5}{(q)_\infty}$$

We have

Theorem For $n \geq 1$,

$$(*) \qquad a_5(n-1) = \sum_{d|n} \left(\frac{d}{5}\right) \frac{n}{d}.$$

Corollary For $n \geq 1$,

$$a_5(n-1) = 5^c \prod_{i=1}^{s} \frac{p_i^{a_i+1}-1}{p_i-1} \prod_{j=1}^{t} \frac{q_j^{b_j+1}+(-1)^{b_j}}{q_j+1}$$

where

$$n = 5^c p_1^{a_1} p_2^{a_2} \cdots p_s^{a_s} q_1^{b_1} q_2^{b_2} \cdots q_t^{b_t}$$

is prime factorization of n where the p_i are primes $\equiv 1, 4 \pmod 5$ & q_j are primes $\equiv 2, 3 \pmod 5$.

Proof This follows from the fact that the function on the rhs $(*)$ is a multiplicative function of n.

Corollary For $n \geq 0$,

$$a_5(5n+4) = a_5(n).$$

Proof $a_5(5n-1) = 5\, a_5(n-1)$ for $n \geq 1$.

So $a_5(5(n+1)-1) = 5 a_5(n)$

& $a_5(5n+4) = 5 a_5(n)$. $\square$

Corollary $\displaystyle\sum_{n=0}^{\infty} p(5n+4) q^n = 5 \prod_{n=1}^{\infty} \frac{(1-q^{5n})^5}{(1-q^n)^6}$.

Proof $$\sum_{n=0}^{\infty} p(n) q^n = \frac{1}{(q)_\infty} = \frac{(q^5;q^5)_\infty^5}{(q)_\infty} \cdot \frac{1}{(q^5;q^5)_\infty^5}$$

Hence $$\sum_{n=0}^{\infty} p(5n+4) q^{5n+4} = \frac{1}{(q^5;q^5)_\infty^5} \sum_{n=0}^{\infty} a_5(5n+4) q^{5n+4}$$

$$\sum_{n=0}^{\infty} p(5n+4) q^{5n} = \frac{1}{(q^5;q^5)_\infty^5} \sum_{n=0}^{\infty} a_5(5n+4) q^{5n}$$

$$\sum_{n=0}^{\infty} p(5n+4) q^{n} = \frac{1}{(q)_\infty^5} \sum_{n=0}^{\infty} a_5(5n+4) q^{n}$$

$$= \frac{1}{(q)_\infty^5} 5 \sum_{n=0}^{\infty} a_5(n) q^n$$

$$= \frac{5}{(q)_\infty^5} \frac{(q^5;q^5)_\infty^5}{(q)_\infty}$$

$$= 5 \frac{(q^5;q^5)_\infty^5}{(q)_\infty^6} = 5 \prod_{n=1}^{\infty} \frac{(1-q^{5n})^5}{(1-q^n)^6}. \qquad \square$$

Chapter 6 - References

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