Congruences for weight 3/2 eta-quotients and their connection with mod 4 conjectures for the spt function and unimodal sequences

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ABSTRACT - JOINT WORK WITH RONG CHEN

- Recently the speaker and Rong Chen (Shanghai) proved Bryson, Ono, Pitman and Rhoades's mod 4 conjectures for strongly unimodal sequences and Lim, Kim and Lovejoy's mod 4 conjectures for odd-balanced unimodal sequences as well as some mod 4 conjectures for the Andrews spt function.
- In this talk we show how we found a similar mod 4 behaviour for certain weight 3/2 eta-quotients. This led to a connection with the Hurwitz class number and eventually gave us the clue for solving the mod 4 unimodal sequence conjectures.
- This is joint work with Rong Chen (Shanghai).

THE PARTITION FUNCTION

Let $p(n)$ denote the number of partitions of n.

Example The partitions of 4 are

$$4, \quad 3+1, \quad 2+2, \quad 2+1+1, \quad 1+1+1+1,$$

so that $p(4) = 5$.

```
> with(qseries);
```

$$[J2jaclist, Jetamake, Jterm2JACPROD, aqprod, briefqshelp, \ldots$$

$$\vdots$$

$$tripleprod, winquist, zqfactor]$$

```
> etaq(q,1,1000);
```

$$1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + q^{22} + q^{26} + \cdots + q^{852} + q^{876} - q^{925} - q^{950}$$

```
> P:=series(1/etaq(q,1,1000),q,1001);
```

$$1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + 11q^6 + 15q^7 + 22q^8 + \cdots + \cdots + 24061467864032622473692149727991q^{1000}$$

```
> findcong(P,1000);
```

$$[4, 5, 5]$$
$$[5, 7, 7]$$
$$[6, 11, 11]$$
$$[24, 25, 25]$$

RAMANUJAN

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5}, \\ p(7n+5) &\equiv 0 \pmod{7}, \\ p(11n+6) &\equiv 0 \pmod{11} \\ p(25n+24) &\equiv 0 \pmod{25} \\ &\vdots \end{aligned}$$

SPT function

- Andrews (2008) defined the function $\mathrm{spt}(n)$ as the total number of appearances of the smallest parts in the partitions of n. For example,

$$\dot{4}, \quad 3+\dot{1}, \quad \dot{2}+\dot{2}, \quad 2+\dot{1}+\dot{1}, \quad \dot{1}+\dot{1}+\dot{1}+\dot{1}.$$

  Hence, $\mathrm{spt}(4) = 10$.

- $$\sum_{n=1}^{\infty} \mathrm{spt}(n) q^n = \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2(1-q^{n+1})(1-q^{n+2})\cdots}$$

$$\sum_{n=1}^{\infty} \mathrm{spt}(n) q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^{n(n+1)/2}(1+q^n)(1-q^{n^2})}{(1-q^n)^2}$$

```
> with(qseries):
> SPTG:=series(1/etaq(q,1,1001)*
add( (-1)^(n-1)*q^(n*(n+1)/2)*(1-q^(n^2))*(1+q^n)
/(1-q^n)^2,n=1..46),q,1001);
```

$$q + 3q^2 + 5q^3 + 10q^4 + 14q^5 + 26q^6 + 35q^7 + 57q^8 + 80q^9 + \\ + \cdots + 6006565709578822481557464728362 74q^{1000} + O(q^{1001})$$

```
> with(qsOEIS);

      [getOEISseq, grabOEIS, matchOEIS, qs2L,
   qsOEISchanges, qsOEISpversion, seqlist2string]

> L:=qs2L(SPTG,1,19);

L := [1, 3, 5, 10, 14, 26, 35, 57, 80, 119, 161, 238,
        315, 440, 589, 801, 1048, 1407, 1820]

> matchOEIS(L);
There were 1 matches (returning the first 1)
92269, "Spt function:  total number of smallest parts
(counted with multiplicity) in all partitions of n."
```

```
> findcong(SPTG,1000);
```

$$[4, 5, 5]$$
$$[5, 7, 7]$$
$$[6, 13, 13]$$
$$[4, 25, 2]$$
$$[9, 25, 4]$$
$$[14, 25, 4]$$
$$[19, 25, 2]$$

ANDREWS (2008) proved that

$$\begin{aligned} spt(5n+4) &\equiv 0 \pmod{5}, \\ spt(7n+5) &\equiv 0 \pmod{7}, \\ spt(13n+6) &\equiv 0 \pmod{13}. \end{aligned}$$

ALSO

$$\begin{aligned} spt(25n+4) &\equiv 0 \pmod{2}, \\ spt(25n+9) &\equiv 0 \pmod{4}, \\ spt(25n+14) &\equiv 0 \pmod{4}, \\ spt(25n+19) &\equiv 0 \pmod{2}, \\ &\vdots \end{aligned}$$

SPT mod 2 and 4

PARITY OF SPT
[FOLSOM and ONO (2008)]
[ANDREWS, G. and LIANG (2011–2013)]

$\mathrm{spt}(n)$ is odd and if and only if $24n - 1 = p^{4a+1}m^2$ for some prime $p \equiv 23 \pmod{24}$ and some integers a, m, where $(p, m) = 1$.

SPT MOD 4
[RONG CHEN and G. (2021)]

For $n > 0$ be an integer, $\mathrm{spt}(n) \equiv 2 \pmod 4$ if and only if $24n - 1$ has the form

$$24n - 1 = p_1^{4a+1} p_2^{4b+1} m^2,$$

where p_1 and p_2 are primes such that $\left(\frac{p_1}{p_2}\right) = -\varepsilon(p_2)$ for $\varepsilon(p) = -1$ if $p \equiv \pm 5 \pmod{24}$ and $\varepsilon(p) = 1$ otherwise, $(m, p_1 p_2) = 1$ and $a, b \geq 0$ are integers.

COROLLARY

Let $p > 3$ be a prime where $p \not\equiv 23 \pmod{24}$. Suppose $24\delta_p \equiv 1 \pmod{p^2}$, $k, n \in \mathbb{Z}$ and $\left(\dfrac{k}{p}\right) = \varepsilon(p)$ where $\varepsilon(p) = -1$ if $p \equiv \pm 5 \pmod{24}$ and $\varepsilon(p) = 1$ otherwise. Then

$$\mathrm{spt}(p^2 n + (pk+1)\delta_p) \equiv 0 \pmod{4}.$$

A sequence of integers $\{a_j\}_{j=1}^{s}$ is a **strongly unimodal sequence** of size n if it satisfies

$$0 < a_1 < a_2 < \cdots < a_k > a_{k+1} > \cdots > a_s > 0 \quad \text{and} \quad a_1 + a_2 + \cdots + a_s = n,$$

for some k. Let $u(n)$ be the number of such sequences.
EXAMPLE: $n = 5$:

$$0 < 1 < 4 > 0$$
$$0 < 1 < 3 > 1 > 0$$
$$0 < 2 < 3 > 0$$
$$0 < 3 > 2 > 0$$
$$0 < 4 > 1 > 0$$
$$0 < 5 > 0$$

$u(5) = 6$

GENERATING FUNCTION:

$$\mathcal{U}(q) := \sum_{n} u(n)q^n$$

$$= \sum_{n=0}^{\infty}(1+q)(1+q^2)\cdots(1+q^n)q^{n+1}(1+q^n)\cdots(1+q^2)(1+q)$$

$$= q + q^2 + 3\,q^3 + 4\,q^4 + 6\,q^5 + 10\,q^6 + 15\,q^7 + 21\,q^8$$

$$+ 30\,q^9 + 43\,q^{10} + 59\,q^{11} + 82\,q^{12} + 111\,q^{13} + 148\,q^{14} + \cdots$$

BRYSON, ONO, PITMAN, RHOADES CONJ.(2012)
CHEN and G. (2021)

Suppose $\ell \equiv 7, 11, 13, 17 \pmod{24}$ is prime and $\left(\frac{k}{\ell}\right) = -1$. Then for all n we have

$$u(\ell^2 n + kl - s(\ell)) \equiv 0 \pmod{4},$$

where $s(\ell) = \frac{1}{24}(\ell^2 - 1)$.

A sequence is called **odd-balanced** if the peak is even, even parts to the left and right of the peak are distinct and the odd parts to the left of the peak are identical with those to the right. We let $v(n)$ be the number of odd-balanced unimodal sequences of size $2n+2$.

THE GENERATING FUNCTION

$$\mathcal{V}(q) := \mathcal{V}(1;q) = \sum_n v(n)q^n = \sum_{n=0}^{\infty} \frac{(-q;q)_n(-q;q)_n q^n}{(q;q^2)_{n+1}}$$

$$= 1 + 2\,q + 5\,q^2 + 9\,q^3 + 16\,q^4 + 29\,q^5 + 48\,q^6 + 77\,q^7 + 123\,q^8 + 191\,q^9 + 290\,q^{10} + 436\,q^{11} + 643\,q^{12} + 936\,q^{13} + 1352\,q^{14} + \cdots$$

where $(a)_\infty = (a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n)$.

KIM, LIM, LOVEJOY CONJ.(2016)
CHEN and G. (2021)

Let $p \not\equiv \pm 1 \pmod{8}$ be an odd prime, suppose $8\delta_p \equiv 1 \pmod{p^2}$ and $k, n \in \mathbb{Z}$ where $\left(\frac{k}{p}\right) = 1$. Then

$$v(p^2 n + (pk - 7)\delta_p) \equiv 0 \pmod{4}.$$

WEIGHT 3/2 ETA-PRODUCTS

The SEARCH for similar congruences in the theory of modular forms.
We define

$$a(n) = \text{the number of representations of } n \text{ as a sum of two pentagonal and three times a triangular number,}$$

$$b(n) = \text{the number of representations of } n \text{ as a sum of a pentagonal and three times the sum of two triangular numbers,}$$

$$c(n) = \text{the number of representations of } n \text{ as a sum of a pentagonal andtwo triangular numbers,}$$

so that

$$\sum_{n=0}^{\infty} a(n)q^n = \left(\sum_{k=-\infty}^{\infty} q^{k(3k+1)/2}\right)^2 \sum_{m=0}^{\infty} q^{3m(m+1)/2} = \frac{J_3^3 J_2^2}{J_1^2} = q^{-11/24}\frac{\eta(3\tau)^3\eta(2\tau)^2}{\eta(\tau)^2},$$

$$\sum_{n=0}^{\infty} b(n)q^n = \sum_{k=-\infty}^{\infty} q^{k(3k+1)/2} \left(\sum_{m=0}^{\infty} q^{3m(m+1)/2}\right)^2 = \frac{J_6^3 J_2}{J_1} = q^{-19/24}\frac{\eta(6\tau)^3\eta(2\tau)}{\eta(\tau)},$$

$$\sum_{n=0}^{\infty} c(n)q^n = \sum_{k=-\infty}^{\infty} q^{k(3k+1)/2} \left(\sum_{m=0}^{\infty} q^{m(m+1)/2}\right)^2 = \frac{J_3^2 J_2^5}{J_6 J_1^3} = q^{-7/24}\frac{\eta(3\tau)^2\eta(2\tau)^5}{\eta(6\tau)\eta(\tau)^3}.$$

Here we have used the usual notation for infinite products and the Dedekind eta-function

$$J_k = \prod_{n=1}^{\infty}(1-q^{kn}), \qquad \eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n),$$

where $q = \exp(2\pi i\tau)$ and $\Im(\tau) > 0$.

$$\sum_{k=-\infty}^{\infty} q^{k(3k+1)/2} = \frac{J_3^2 J_2}{J_6 J_1}, \qquad \sum_{k=0}^{\infty} q^{k(k+1)/2} = \frac{J_2^2}{J_1}$$

ETA HAS WEIGHT $1/2$

$$\eta(\tau+1) = \exp(2\pi i/12)\,\eta(\tau), \qquad \eta(-1/\tau) = \sqrt{-i\tau}\,\eta(\tau)$$

[RONG CHEN and G. (2021)]

Let $p > 3$ be prime, suppose $24\delta_p \equiv 1 \pmod{p^2}$, and k, $n \in \mathbb{Z}$ where $\left(\frac{k}{p}\right) = 1$. Then

$$a(p^2n + (pk - 11)\delta_p) \equiv 0 \pmod{4}, \qquad \text{if } p \not\equiv 11 \pmod{24},$$
$$b(p^2n + (pk - 19)\delta_p) \equiv 0 \pmod{4}, \qquad \text{if } p \not\equiv 19 \pmod{24},$$
$$c(p^2n + (pk - 7)\delta_p) \equiv 0 \pmod{4}, \qquad \text{if } p \not\equiv 7 \pmod{24}.$$

THE SEARCH for weight 3/2 eta-quotients with ...

Suppose $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty} \subset \mathbb{Z}$ and

$$1 + \sum_{n=1}^{\infty} a_n q^n = \prod_{n=1}^{\infty} (1 - q^n)^{b_n},$$

holds formally. Then

$$n a_n = -\sum_{j=1}^{n} D_j a_{n-j}, \qquad \text{where } D_j = \sum_{d|j} d b_d \tag{i}$$

$$b_n = a_n - \frac{1}{n} \left(\sum_{\substack{d|j \\ d<n}} d b_d + \sum_{j=1}^{n-1} D_j a_{n-j} \right) \tag{ii}$$

prodmake

```
> with(qseries):
> A:=series(exp(q),q,20);
```

$$1 + q + \frac{1}{2}q^2 + \frac{1}{6}q^3 + \frac{1}{24}q^4 + \frac{1}{120}q^5 + \cdots + \frac{1}{121645100408832000}q^{19} + O\left(q^{20}\right)$$

```
> prodmake(A,q,20,list);
```

$$[-1, \frac{1}{2}, \frac{1}{3}, 0, \frac{1}{5}, -\frac{1}{6}, \frac{1}{7}, 0, 0, -\frac{1}{10}, \frac{1}{11}, 0, \frac{1}{13}, -\frac{1}{14}, -\frac{1}{15}, 0, \frac{1}{17}, 0, \frac{1}{19}]$$

$$\exp(q) = \prod_{n=1}^{\infty} (1 - q^n)^{-\mu(n)/n}$$

eta-quotients and etamake

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n),$$

where $q = \exp(2\pi i \tau)$ and $\Im(\tau) > 0$.

$$f(\tau) = \prod_{d|N} \eta(d\tau)^{m_d},$$

where N is a positive integer and each $d > 0$ and $m_d \in \mathbb{Z}$.

etamake(f,q,T) — attempts to convert f to an eta-quotient up to q^T

```
> with(qseries):
> T3:=add(q^(n^2),n=-10..10);
```

$$2\,q^{100}+2\,q^{81}+2\,q^{64}+2\,q^{49}+2\,q^{36}+2\,q^{25}+2\,q^{16}+2\,q^{9}+2\,q^{4}+2\,q+1$$

```
> etamake(T3,q,100);
```

$$\frac{\eta(2\tau)^5}{\eta(4\tau)^2\,\eta(\tau)^2}$$

Searching for weight $3/2$ eta-quotients with . . . Let $1 \leq \ell < 24$ where $(\ell, 24) = 1$ and

$$F = \sum_{n=0}^{\infty} f(n) q^{n+\ell/24} = \prod_{d|12} \eta(d\tau)^{m_d}.$$

We define a function **rescheck(F, p, ℓ)** which returns a pair $[\alpha_1, \alpha_2]$ such that

$$f(p^2 n + (pk - \ell)/24) \equiv 0 \pmod{2^{\alpha_1}}, \quad \text{for } \left(\frac{k}{p}\right) = 1,$$

and

$$f(p^2 n + (pk - \ell)/24) \equiv 0 \pmod{2^{\alpha_2}}, \quad \text{for } \left(\frac{k}{p}\right) = -1,$$

where p is prime and α_1 and α_2 are DISTINCT postive integers.

We define a function **esearch(T,p,ℓ)** which checks all eta-quotients of level 12 with bounded exponents for desired α_1, α_2 for a given prime *p* not congruent to ℓ (mod 24).

```
> EL1:=esearch(2,5,7):
> nops(EL1);
```

6

```
> etamake(EL1[1],q,20);
```

$$\frac{(\eta(6\tau))^4 (\eta(4\tau))^2}{(\eta(12\tau))^2 \eta(\tau)} q^{-\frac{7}{24}}$$

```
> PRIMES:=[seq(ithprime(j),j=3..22)]:
> K:=seq([p,rescheckv2(EL1[1],p,7)],p in PRIMES);
```

$$K := [5,[1,2]],[7,[0,1]],[11,[1,2]],[13,[2,1]],[17,[2,1]],\\ [19,[2,1]],[23,[2,1]],[29,[1,2]],[31,[0,1]],[37,[2,1]],\\ [41,[2,1]],[43,[2,1]],[47,[2,1]],[53,[1,2]],[59,[1,2]],\\ [61,[2,1]],[67,[2,1]],[71,[2,1]],[73,[2,1]],[79,[0,1]]$$

CONJECTURE Let $p > 3$ be prime. Define

$$\sum_{n=0}^{\infty} f(n) q^{n+7/24} = \frac{(\eta(6\tau))^4 (\eta(4\tau))^2}{(\eta(12\tau))^2 \eta(\tau)}.$$

Then

$$f(p^2 n + (pk-7)/24) \equiv 0 \pmod 2, \quad \text{for } \left(\frac{k}{p}\right) = 1 \text{ and } p \equiv \pm 5\,(24),$$

and

$$f(p^2 n + (pk-\ell)/24) \equiv 0 \pmod 4, \quad \text{for } \left(\frac{k}{p}\right) = -1 \text{ and } p \not\equiv 7, \pm 5\,(24).$$

findETAcongs(ℓ, T, T2, NPL)

ℓ – prime representing a residue class mod 24

T,T2 – two positive integers

NPL – integer > 2

- Find [a, b, c] such that

$$ax^2 + by^2 + cz^2 = \ell$$

- Find $d\, f(n) = r_{a,b,c}(24n + \ell)$ where

$$\sum_{n=0}^{\infty} r_{a,b,c}(n)q^n = \sum_x \sum_y \sum_z q^{ax^2+by^2+cz^2}.$$

- Use **prodmake** and **etamake** to check whether $\sum_{n \leq T} f(n) q^n$ is a likely eta-quotient, and if so compute up to q^{T2}.
- Use the function **rescheck** to check for congruences of the form

$$f(p^2 n + (pk - \ell)/24) \equiv 0 \pmod{2^\alpha},$$

for all n, and that depend on $\left(\frac{k}{p}\right) = 1$ (within limits).


Questions ?


Thank You!

THANK YOU

REFERENCES

- George E. Andrews, *The number of smallest parts in the partitions of n*, J. Reine Angew. Math. **624** (2008), 133–142.
- R. Chen and F. G. Garvan., *Congruences modulo* 4 *for weight* 3/2 *eta-products*, Bull. Austral. Math. Soc., doi:10.1017/S0004972720000982, to appear.
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